A New Method of Calculating Optimum Velocity Distribution Along the Blade Surface on Arbitrary Stream Surface of Revolution in Turbomachines

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ABSTRACT:
This paper presents a physical model and its mathematical expressions (partial differential equation group), which are to be used to calculate the optimum velocity distribution on blade surface. The method is based on the theory of boundary layer and the calculation of cascade loss, and to employ the Pontryagin maximum principle as well as the new optimum techniques in applied mathematics. In this paper, a computing method of optimum velocity distribution along the blade surface in 2-D incompressible flow is presented by analysing and solving the equation group, and then by using the method which is presented by Zou Zixiang (1976), and through a logical analysis, a new method has been offered, which can converted from an optimum velocity distribution along the plane stream surface of incompressible fluid flow into that of an arbitrary stream surface of revolution in compressible fluid flow.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>velocity of relative flow</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>= $w/W_0$; $W_0$ is inlet velocity</td>
</tr>
<tr>
<td>$\theta$</td>
<td>= momentum thickness</td>
</tr>
<tr>
<td>$H_{12}$</td>
<td>= $H_{12} = \delta^*/\theta$</td>
</tr>
<tr>
<td>$H_{32}$</td>
<td>= $H_{32} = e/\theta$</td>
</tr>
<tr>
<td>$\delta^*$</td>
<td>= displacement thickness</td>
</tr>
<tr>
<td>$e$</td>
<td>= energy thickness</td>
</tr>
<tr>
<td>$\delta_{tn}$</td>
<td>= thickness of outer edge</td>
</tr>
<tr>
<td>$\Delta \phi$</td>
<td>= angle pitch</td>
</tr>
<tr>
<td>$\beta$</td>
<td>= angle of relative flow</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>= radius of rotation</td>
</tr>
<tr>
<td>$\rho$</td>
<td>= density</td>
</tr>
<tr>
<td>$\tau_w$</td>
<td>= local tangential strength</td>
</tr>
<tr>
<td>$F$</td>
<td>= Buri rule function or form function</td>
</tr>
<tr>
<td>$R_{0}$</td>
<td>= $R$-number based on the momentum thickness</td>
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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>$T$</td>
<td>= absolute temperature</td>
</tr>
<tr>
<td>$a$</td>
<td>= local sound velocity</td>
</tr>
<tr>
<td>$L$</td>
<td>= arc length from inlet stagnation point to outer edge point</td>
</tr>
<tr>
<td>$K$</td>
<td>= adiabatic index</td>
</tr>
<tr>
<td>$\tau$</td>
<td>= normal thickness of stream sheet on arbitrary stream surface of revolution</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>= velocity circulation of blade surface</td>
</tr>
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SUBSCRIPTS

1 or e | = parameters at outer edge
0 | = parameters at inlet or stagnation point
$Sgn$ | = representant the signal (plus or minus)
f | = free flow at outer boundary of boundary layer
t | = pitch of cascade
r | = compressible flow on stream surface of revolution
tn | = normal thickness at outer edge

INTRODUCTION

The optimum design of cascade means that the loss of kinetic energy or total pressure of the designed cascade will be smallest possible in given conditions. This is a difficult problem, because there exists a very complicated relation between all the parameters. It is almost impossible to find out a mathematical expression in normal calculus of variations. It is a proposition of "Phase" and "Control" constraint with multi-inequalities in mathematics. The normal calculus is not very much helpful in dealing with this problem. In recent years the advance in applied mathematics, especially the optimum control theory, has made it easier for us to solve this problem. A general description of cascade
optimum design theory in 2-D incompressible flow was summarized by Liu GaoLIN (1977). From the papers of Papailiou (1971), Citavy (1974), and Shuang Huo, (1972), we can find a simplified method of calculation of the cascades in 2-D compressible flow. The experiments showed that the results was good. The physical model in this paper is to simplify the complicated problem by first breaking up it into a series of sub-problems, in which there exist weak relations between these sub-problems, and then solving them respectively. In other words, we first calculate the optimum velocity distribution along the suction side of blade profile, and then obtain the profile of the blade which must satisfy the given conditions, i.e. the above-mentioned optimum velocity distribution and the thickness distribution of the blade section, and which must also satisfy the constraints of strength, vibration, cooling, and technology, etc.. A closed end of the blade section is a prerequisite for a proper determination of the velocity on the pressure side. This determination is, of cause, not optimal. In this paper we put emphasis on the above-mentioned first step, and we have studied a few methods of calculation, such as the methods of boundary layer presented by Buri,of cascade loss shown by Stewart (1955) and in my paper (1976), etc.. The modern calculus of variations including the famous Pontryagin maximum principle, and the 'principle of constraining separate' as well as the 'principle of sectional maximum' in optimum technique, also were used. The optimum velocity distribution on blade surface in 2-D incompressible steady flow is first calculated, and then it is extended to that of the compressible flow on an arbitrary surface of revolution. Moreover, it is still true that the transformed optimum velocity distribution remains at its optimum. The methods presented in this paper can also be applied to the optimum design of two-dimensional compressor blades, as well as long thin revolution bodies of axial symmetry.

OPTIMUM VELOCITY DISTRIBUTION ON BLADE SURFACE IN TWO-DIMENSIONAL INCOMPRESSIBLE FLOW

The Proposition of the Problem

Is to obtain an optimum velocity distribution on the blade suction side for the given conditions: inlet and outlet angle $\beta_1, \beta_2$; stage angle $\beta_3$, the velocity circulation and boundary layer's inseparable condition. The kinetic energy loss or total pressure loss of the designed blade cascade, which corresponds to this velocity distribution, must be smallest possible. The analysis of cascade loss Stewart (1955) and myself (1976) have proved that a minimum energy loss coefficient of cascade $c_{min}$ happens simultaneously with a minimum momentum thickness $\theta_{min}$ at the outer edge of the blade. The above-mentioned proposition of the optimum problem can be converted into the calculating of the minimum momentum thickness $\theta_{min}$ and both the results are equivalent.


Assumptions: the flow fluid is Newton's fluid with viscosity; the flow is adiabatic, steady, compressible and the flow in boundary layer is full turbulent. Based on these assumptions, we get following governing equations, which are the momentum integral equation of incompressible turbulent boundary layer, Buri's form function, and the differential equation of the target function $\xi$, which is the additional state variable.

$$\frac{d\theta}{dx} + (H_n + 2) \left( \frac{dW_f}{W_f} \right) = \frac{C_w}{\beta} \frac{W_f^2}{\rho} \quad (1)$$

$$\frac{dW_f}{dx} = \left( \frac{W_f}{\rho} \right) Re \frac{W_f}{\theta} \quad (2)$$

$$\frac{dS}{dx} = \left( H_n + 2 \right) \left( \frac{dW_f}{W_f} \right) \frac{dW_f}{dx} \quad (3)$$

Equation (3) was derived from equation (22) presented by Stewart (1955). We suppose that the momentum thickness of pressure side is much smaller than that of the suction side. Let $x=0$ then $\xi=0$; So the Payoff Function $S$, which is to be minimized, can be written: $S = \xi(x_L)$ where the $x_L$ represents the value of outer edge terminal point. If we want $\Gamma > 0$ (mk=cons.) in the condition of $S$ approaching minimum, the change of $S$ will be greater than 0, i.e. $\Delta S > 0$. The necessary condition of $\Delta S > 0$ is to satisfy $\Delta H > 0$, i.e. the change of Hamilton function $H$ is caused by the change of control variable $\Gamma$, such as $\Delta H$ must be smaller than zero, $\Delta H < 0$.

According to our specific situations, Hamilton Function can be represented as follows:

$$H = \sum_{i=1}^{N} \left( H_i (W_f, \theta) Re \frac{W_f}{\theta} \right)$$

where the coefficient of control variable $\Gamma$ is called switch function $K$:

$$K = \sum_{i=1}^{N} \left( H_i (W_f, \theta) Re \frac{W_f}{\theta} \right)$$

$P_1, P_2, ..., P_i$ are additional variables in 'Pontryagin maximum principle', their differential equation group is commensurate with Euler equation group in calculus of variations:

$$P_i = \sum_{i=1}^{N} P_i \left( \frac{\partial f_i}{\partial \xi_i} \right) \quad i=1, ..., N=3 \quad (6)$$

To satisfy the minimum necessary condition $\Delta H < 0$, from equation (4), we know: when $K>0$ than $\Gamma_{min}$; when $K<0$ than $\Gamma_{max}$; when $K=0$, $\Gamma$ cannot get a control effect, it is called singular solution. The sign of $\Gamma$ is defined by the sign of $K$: $\Gamma = \text{sign}(K)$.

In equations (1-2), $W_f, \theta, \xi$ are called state variables, their initial conditions are $W_f(0)=0; \theta(0)=0, \xi(0)=0$. $\Gamma$ is called control variable. According to the reference, boundary layer's inseparable condition is limited in $\Gamma > 0.04$, another velocity circulation constraint is presented by $\frac{dW_f}{dx} = \frac{dW_f}{dx} \Gamma$; in the calculating procedure, let the outlet velocity equal to the exhaust velocity: $W_f(X_L) = W_f$. As above analysis, we can get eight equations (Equation 6 consists of three equations), and there are eight unknown variables $W_f, \theta, \xi, K, H, P_1, P_2, P_3$ in the equations, so that the equations are closed.

To our specific situation, the sign of $K$ can be obtained by analysis. Therefore, it is not necessary to calculate the additional variables $P_i$ only to calculate equations (1-3).

To Solve the Differential Equation Group:

Combine (1) with (2), we get

$$\frac{d\theta}{dx} + (H_n + 2) \left( \frac{dW_f}{W_f} \right) = \frac{C_w}{\beta} \frac{W_f^2}{\rho} \quad (7)$$
when the 'Principle of Constraining separate' is used, equation (7) shows to us that if \( \Gamma \) is equal to critical limit, there will be one part of solution, and if greater than critical limit, there is another part of solution. The solution consists of two parts:

a). \( \Gamma = mk \) (boundary layer's inseparable constant). According to the assumption given by Buri, function \( r_{\alpha L} \) and \( H_{12} \) are the only responses to the form function \( \Gamma \), thus (7) can be expressed as:

\[
\frac{d(\theta R_{\alpha L})}{dx} = 5/4 \left[ f'(r) - \Gamma (a''_L + H_1) \right]
\]

Buri further found that \( A-B \) can be approximately considered as a linear function, so (8) becomes

\[
\frac{d(\theta R_{\alpha L})}{dx} = A-B \Gamma
\]

If \( \Gamma \) is equal to constant (A=0.016, B=4, \( H_{12}=1.4 \)), (9) becomes an easier integral equation, therefore,

\[
\theta = R_{\alpha L} \cdot D (x-x_L) + \theta (x_L) R_{\alpha L} x_{\alpha L}^{-1/4}
\]

\[
W_f = W_f (x_L) e^{(7/10) \int [(x-x_L) + (x_0-x)] / (x_0-x)]}
\]

where \((A-B) \Gamma = A = \theta (x_L) R_{\alpha L} (x_L) \). When \( x=x_L \), the additional constraint condition \( W_f (x_L)=W_f^{(L)} \) must be satisfied. \( x_L \) is an integral initial point, it is the intersection point joining the two parts of solution.

b). \( \Gamma > mk \) (means that \( \Gamma \) is not constrained). When the \( \Gamma \) is not constrained, it can be considered as a normal calculus of variations, i.e. to calculate the calculus of variations of \( \theta_{1\text{min}} \). From (9) and (2), we get:

\[
\frac{d(\theta R_{\alpha L})}{dx} + [B(\theta R_{\alpha L})/W_f] dW_f/dx = A
\]

It is a linear ordinary differential equation about \( \theta R_{\alpha L} \), easy to be integrated:

\[
W_f^0 \theta R_{\alpha L}^A = A \int_{x_0} x_{\alpha L} W_f^B dX + \text{const}
\]

To expand \( R_{\alpha L} \), the momentum thickness is

\[
G_{1/4} = (x_{\alpha L} / h_f^{\alpha L}/4) (A \int_{x_0} x_{\alpha L} W_f^B dX + \text{const})
\]

and

\[
(G_{1/4} + C_4) = C_4 \int_{x_0} x_{\alpha L} W_f^B dX
\]

thus, the calculation of the \( \theta_{1\text{min}} \) is converted into the calculation of the minimum of function

This function's Euler equation is in form of \( \Gamma = 0 \). Obviously, \( W_f^0 \) can have \( W_f^{(L)} \) reached minimum. With the added constraint of \( \Gamma \), we get the optimum distribution on blade surface is \( W_f^{\text{opt}}=\text{const} \). and \( W_f^{\text{opt}}=\phi_{\text{opt}} \).

c). Optimum Velocity Distribution at Various Different Conditions: (1) If the initial and final velocity is given, and the constraint of boundary layer's inseparability is added, various optimum velocity distributions are all shown in Fig.1 where \( a'_L \), \( a''_L \), \( a''''_L \) respectively correspond to optimum velocity distribution of \( \phi_0 \), \( \phi_0^* \), \( \phi_0^{**} \). If \( \phi \) lies between \( \phi_0 \) and \( \phi_0^* \), the optimum velocity distribution, for instance the \( a''''_L \), consists of three parts: 'descending-flat-descending'; and, if \( \phi \) ranges from \( \phi_0^* \) to \( \phi_0^{**} \), it consists of three parts of 'descending-flat-upright'; If \( \phi \) is between \( \phi_0 \) and \( \phi_{kp} \), it is made up of two parts of 'flat-upright'. Here, \( \phi_0 \) represents the smaller between \( \phi_{\text{max}} \) and \( \phi_{\text{M}} \), the flat part of \( \phi_{\text{max}} \) is just equal to the velocity circulation when \( W_f^{\text{max}} \) is in 'two parts distribution.' (2) If the initial velocity is unknown, but the final velocity is given, the optimum velocity distribution is discussed as follows: when lies between \( \phi_{Kp} \) and \( \phi_{M} \), the optimum distribution is only \( a''_L \) in Fig.2, but is not the three parts type of \( a''_L \) or \( a''''_L \). Up to now, we have studied the components of various optimum velocity distributions, and get the constraint of velocity circulation, i.e. if the initial velocity is given, then \( \phi_{\text{opt}} = \phi_0 \), if it is not given, then \( \phi_{\text{opt}} = \phi^{**}_0 \). Here, \( \phi_{\text{opt}} = \phi_{\text{min}} \). In order to improve the condition of flow on nose of the blade, a transitional section from \( 0 \) to \( X_0 \) is added in Fig.2. Meanwhile, to have the velocity at \( X_0 \) being smooth, we select sin distribution in the range of \((0, X_0/2)\), so the maximum from stagnation point \( 0 \) rising to \( X_0 \) is \( W_f^{(L)}=W_f^{\text{max}} \.

\[
\text{Fig.1 With inseparable constraint}
\]

\[
\text{Fig.2 The initial velocity is not given}
\]

OPTIMUM VELOCITY DISTRIBUTION ON BLADE SURFACE OF COMPRESSIBLE FLOW ON THE ARBITRARY SURFACE OF REVOLUTION

By using the transformed method expressed by Ager (1958), Loichanski, Zou Zixiang (1976), the boundary layer parameters and velocity distribution of 2-D incompressible flow can be converted into that of arbitrary surface of revolution of compressible flow, and then we have

\[
\gamma = \gamma (x) \quad \phi = \partial x / \partial (x_0) \quad \theta = \theta / \partial (x_0) \quad H_0 = H_0 \quad W_f = W_f \quad M_f = M_f / \gamma \quad a_0 = a_0 / \beta \quad \phi = \phi / \gamma \quad \theta = \theta / \gamma \quad H_0 = H_0 / \gamma
\]
where the $\tau(x)$ is normal thickness of stream sheet, the sign" represents 2-D compressible flow parameters, while the sign' represents incompressible ones.

In the following part of the paper, we are going to prove that the optimum velocity distribution on the blade surface of 2-D incompressible flow, after having been converted into arbitrary surface of revolution, is also the optimum velocity distribution on the blade surface.

By analysing and studying the calculating method of cascade loss coefficient and momentum thickness expressed in the papers of Stewart (1955) and Zou Zixiang (1976), it has been proved that the minimum loss coefficient of energy on blade of 2-D incompressible flow happens at the same time with the minimum of momentum thickness at outlet boundary layer $\partial j$ and similarly we have the other responding parameters whose minimum appears simultaneously:

2-D compressible $\xi''$ with $\xi''$

and arbitrary revolutionary surface $\xi$ with $\xi$. By employing the method put forward by Ager (1958), Stewart (1955), Zou Zixiang (1976), we can also prove that the minimum of both $\xi''$ with $\xi''$, as well as $\xi''$ with $\xi''$, appears simultaneously. Since $\eta'' = \xi''$, so $\xi''$ with $\xi''$ and $\xi''$ with $\xi''$ will finally get their minimum at the same time. In the final analysis, after converting the problem of arbitrary surface of revolution the velocity distribution remains optimum velocity distribution on the blade surface. On the other hand, $\xi'' = \xi''$, so $\xi''$ cannot be obtained by transformation, it must be calculated

$$\xi = \frac{a \times \left(1 + (k+1)/2 \right) M^2}{(b - c \times \left(1 + (k+1)/2 \right) M^2) \sin \theta}$$

where $a = (\lambda_{f}/\lambda_{n})^2$, $b = \gamma_{f} \gamma_{e} \cos \theta$, $c = H_{2} \gamma_{e}$, $k$ is adiabatic index.

RESULTS AND DISCUSSION

In the calculated example of this paper, the "flat-descending" two parts type of optimum velocity distribution has been used. The results are show Fig.3 and 4. The momentum thickness distribution obtained according to the optimum velocity distribution is shown in Fig.4. Meanwhile, the corresponding kinetic energy loss coefficient on suction side is $\xi'' = 0.0119$. But in one of my previous papers (1974), the $\xi'' = 0.0138$. It shows that the loss coefficient obtained by using the methods recommended in this paper is smaller than that of any non-optimum designs.

The results are compared with those shown in Citavy's paper (1974), both are similar, they both are of the "flat-descending" two parts type of optimum velocity distribution. But the velocity which at the point $X_{f}$ is discontinuity in Fig.5. It can be also seen from Fig.6 that the momentum thickness obtained according to the optimum velocity distribution is smaller than the measured results shown by N.X. Chen (1977), which is not an optimum designed cascade. It shows that the optimum designed cascade has better efficiency than the non-optimum designed one.
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