Computation of Incompressible Flow in Turbomachines Using the Primitive Variable Formulation

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ABSTRACT

The primitive variable approach is adapted here for the solution of incompressible flow in turbomachines using non-staggered grids. In this approach, a pressure Poisson equation with Neumann boundary conditions is solved in lieu of the continuity equation. Solutions for the Poisson equation exist only if a compatibility condition is satisfied. This condition is not automatically satisfied on non-staggered grids. Failure to satisfy the compatibility condition results in non-convergent solutions. A consistent finite difference method which satisfies this condition using a non-staggered grid in general curvilinear coordinates is developed. Numerical solutions are obtained for the pressure equation using the successive over-relaxation method. The velocity field is computed from the momentum equations by explicitly marching in time. The computed solutions are compared with the available numerical results for both inviscid and viscous laminar flows in cascades.

INTRODUCTION

The development of computational algorithms for compressible flows has advanced rapidly in the past decade. Although the governing equations for incompressible flows seem simpler than those for compressible flows, the computational techniques of the incompressible case are not as well developed.

The major difficulty in the numerical solutions of incompressible flows is computing the pressure from the continuity and momentum equations. At the early stages of development of computational algorithms, the pressure is eliminated from the governing equations as done in the stream function-vorticity and the velocity-vorticity formulations [1]. These procedures, however, result in a large number of differential equations for three-dimensional problems. Thus these techniques are inefficient. There are several techniques that solve a modified form of the Navier-Stokes equations, such as the parabolized and partially-parabolized Navier-Stokes equations [2].

Two formulations are well suitable for the solution of incompressible inviscid and viscous flow equations: the artificial compressibility method and the pressure Poisson equation method. In the artificial compressibility method [3, 4], a time derivative pressure term is added to the continuity equation which allows the use of the compressible flow techniques for the solution of the incompressible case. However, to maintain the accuracy of the fictitious continuity equation during iteration, the artificial compressibility term should be divided by a large factor. Under such situation the system of equations becomes very stiff (ill-conditioned) [5, 6]. Although the implicit schemes can effectively deal with very stiff systems, it still can not deal with the large contamination term (which is proportional to the artificial compressibility factor) introduced by the vectorized ADI procedure [5].

Steger and Kutler [5] first used the artificial compressibility method with Beam-and-Warming ADI technique [7] to compute the vortex wakes. Later Choi and Merkle [8, 9] investigated the stability and convergence of this method for incompressible flows and compared the results with those for the compressible flow solver. Kwak et al. [10, 11] used this method to solve three-dimensional duct flows. Choi and Merkle [8] and Kwak et al. [10, 11] pointed out that if only the steady-state solution is desired, the optimum convergence rate in two-dimensional flows occurs when the artificial compressibility factor is approximately equal to unity. The problems due to stiff eigenvalues can consequently be avoided in such situation. Hence, the artificial compressibility method is limited to steady-state problems. As to applying this formulation to unsteady flow problems, more studies are needed.

The pressure Poisson equation approach solves the unsteady momentum equations for the velocity field and a Poisson equation for the pressure. The pressure equation is derived from the divergence of the momentum equation with the retention of the local dilation term [12]. Instead of being solved directly, the continuity equation in this formulation is satisfied indirectly through the solution of the pressure Poisson equation. Neumann boundary conditions are obtained from the momentum equations applied at the boundaries. Solutions for
a Poisson equation with Neumann boundary conditions exist only if an integral constraint is satisfied. This constraint is a consequence of Green’s first integral theorem [13], and known as the compatibility condition. Harlow and Welch [12] showed that the compatibility condition is automatically satisfied on staggered grids. However, this compatibility condition is not satisfied on non-staggered grids [14]. In addition, the use of staggered grids especially for three-dimensional problems is a major difficulty.

In order to satisfy the compatibility condition on non-staggered grids, two remedies were suggested respectively by Briley [15] and Miyakoda [16]. In Briley’s approach a small uniform correction is added to the source term of the Poisson equation before solving it. The amount of the correction is the average of the error in the compatibility condition obtained by numerical integration over the domain. Ghia et al [17] applied this approach for the solutions of two- and three-dimensional duct flows and concluded that this leads to reasonable results. The method by Miyakoda [16] is less known. He incorporated the Neumann boundary conditions directly into the difference scheme at interior points adjacent to the boundaries. Information about further applications of this method are quite limited.

The problem of the compatibility condition was not actually solved until Abdallah developed a consistent finite difference procedure to satisfy the compatibility condition numerically on non-staggered grids [14, 18]. The principles of this procedure are: (1) The Poisson equation is written in a conservative form, (2) the viscous terms in the momentum equations are expressed in terms of the vorticity, and (3) the Neumann boundary conditions are differenced at half grid point from the boundaries, such that the difference equations are consistent with the difference equations for the Poisson equation at the interior mesh points. He applied the method to the driven cavity problem and obtained very good convergence. Later Abdallah and Smith [19] successfully applied it to three-dimensional inviscid duct flow.

In the present study, the primitive variable approach with a pressure Poisson equation is adopted to the solution of the governing equations of viscous and inviscid flows in generalized curvilinear coordinate system using non-staggered grids. The differencing procedure developed by Abdallah [14, 18, 19] for the solution of the Poisson equation with Neumann boundary conditions is formulated in generalized curvilinear coordinates, such that the compatibility condition is exactly satisfied. The extension to curvilinear coordinates requires careful manipulation of the pressure Poisson equation because of the presence of cross-derivative terms in the generalized form of the pressure equation. Numerical solutions for the pressure Poisson equation are obtained using the successive over-relaxation method. The momentum equations are solved for the velocity field by marching in time explicitly. Only two-dimensional steady flows are investigated in this study, but it is easy to extend the method to unsteady and three-dimensional flows. Details of the procedure are given in the following section.

GOVERNING EQUATIONS

In this section, the incompressible continuity and Navier-Stokes equations are written in Cartesian coordinates, then reformulated in generalized coordinates. In addition, a pressure Poisson equation is derived from the divergence of the momentum equation.

Continuity Equation

\[ u_x + v_y = 0 \] (1)

x-momentum equation

\[ \frac{\partial u}{\partial t} + u u_x + v u_y = -p_x + \frac{1}{Re} (u u_x + u_y) \] (2)

y-momentum equation

\[ \frac{\partial v}{\partial t} + u v_x + v v_y = -p_y + \frac{1}{Re} (v v_x + v_y) \] (3)

where \((u, v)\) are the Cartesian velocity components in \((x, y)\) directions, \(p\) is the static pressure, and \(Re\) is the Reynolds number.

The governing equations (2) and (3) are parabolic in time. They are solved for the velocity components \(u\) and \(v\) by marching in time. The pressure is computed from a Poisson-type equation derived from the divergence of the momentum equation.

Pressure Poisson Equation

By differentiating Eq.(2) w.r.t. \(x\) and Eq.(3) w.r.t. \(y\) and adding, one obtains

\[ p_{xx} + p_{yy} = \frac{\partial D}{\partial t}, \] (4)

where

\[ -\sigma = (u u_x + v u_y) + (u v_x + v v_y) \] (4a)

and

\[ D = u_x + v_y. \] (4b)

Eq.(4) is a second-order elliptic partial differential equation which is solved for \(p\). It should be noted here that the viscous terms in the momentum equations are eliminated from the source term \(\sigma\) by the use of the continuity equation. However, it will be shown later that the viscous terms remain in the boundary conditions for the pressure equation.

Equations in Generalized Coordinates

Eqs.(1), (2), (3), and (4) for the dependent variables \(u, v, \) and \(p\) are not independent. Eq.(1) is eliminated and satisfied through the solution of Eq.(4) [1, 14, 18]. More details about that point is given in the numerical solution section.

The above equations are written in generalized curvilinear coordinate system \(\xi\) and \(\eta\) as follows:

\[ \frac{\partial}{\partial t} u + \left( \frac{U}{J} - \frac{Jr}{Re} \right) u + \left( \frac{V}{J} - \frac{J\theta}{Re} \right) u = -\left( u_{\xi} p_{\xi} - (\xi p_{\xi}) \right) + \frac{J}{Re} \left( a u_{\xi \xi} - 2 \beta u_{\xi \eta} + \gamma u_{\eta \eta} \right), \] (5)

and

\[ \frac{\partial}{\partial t} v + \left( \frac{U}{J} - \frac{Jr}{Re} \right) v + \left( \frac{V}{J} - \frac{J\theta}{Re} \right) v = \left( v_{\xi} p_{\xi} - (\xi p_{\xi}) \right) + \frac{J}{Re} \left( a v_{\xi \xi} - 2 \beta v_{\xi \eta} + \gamma v_{\eta \eta} \right), \] (6)

In the above equations, the parameters are defined as follows:

\[ \frac{1}{J} = \xi_x y_{\eta} - \xi_{\eta} y_x, \] (7a)

\[ U = J(u_{\eta \eta} - v_{\xi \eta}). \] (7b)
\[ V = J(-uy_\xi + vx_\zeta), \]  
\[ \alpha = x_\zeta^2 + y_\zeta^2, \]  
\[ \beta = x_\xi x_\zeta + y_\xi y_\zeta, \]  
\[ \gamma = x_\xi^2 + y_\xi^2, \]  
\[ \theta = J(y_\zeta H_\zeta - x_\zeta H_\xi), \]  
\[ \tau = J(x_\xi H_\zeta - y_\zeta H_\xi), \]

where
\[ H_\xi = \alpha x_\xi - 2\beta x_\zeta + \gamma x_\eta \]  
and
\[ H_\eta = \alpha y_\xi - 2\beta y_\zeta + \gamma y_\eta. \]

The pressure equation, Eq. (4), is written in a divergence form following the method of [1].
\[
\{J(\alpha_\xi - \beta_\eta)\}_\xi + \{J(-\beta_\zeta + \gamma_\eta)\}_\eta = \sigma - \frac{\partial D}{\partial t} \tag{8}
\]

where
\[
\sigma = (y_\eta U_\xi u_\xi + y_\xi V_\eta v_\eta - x_\eta U_\xi v_\xi - x_\xi V_\eta u_\zeta)\xi
+ (y_\xi U_\xi u_\xi + y_\zeta V_\eta v_\eta - x_\xi U_\zeta v_\xi - x_\eta V_\zeta u_\zeta)\eta
\]
and
\[ D = \left(\frac{U}{J}\right)_\xi + \left(\frac{V}{J}\right)_\eta. \tag{8b}\]

The governing equations, Eqs. (5), (6), and (8) may be solved for both inviscid and viscous flows. The inviscid flow equations are obtained by setting \(1/Re = 0\) in Eqs. (5) and (6). In this case, the momentum equations are of the hyperbolic type. The pressure equation, Eq. (8), is identical for both inviscid and viscous flow. The major difference between the two sets of equations for inviscid and viscous flow is in the boundary conditions.

**Boundary Conditions for Velocity**

With reference to Fig. 1, the no-flux condition at the solid boundaries is used to determine boundary conditions for the velocity field for both inviscid and viscous flows.

**Inlet and exit boundary conditions for the velocity field are**
\[
u = 1, \quad \text{and} \quad v = 0 \quad \text{at} \quad AF \tag{9c}
\]
\[
\xi u_\xi + \eta u_\zeta = 0, \quad \text{and} \quad \xi v_\xi + \eta v_\zeta = 0 \quad \text{at} \quad BC \tag{9d}
\]

For inviscid flow solutions, no exit boundary conditions are required since the momentum equations become hyperbolic first order. For viscous flow solutions, the condition of no-slip at the solid boundaries is also required in the solution of the momentum equations, Eqs. (5) and (6).

**Boundary Conditions for Pressure**

The pressure equation, Eq. (8), is elliptic second-order differential equation for both inviscid and viscous flow solutions. Thus, boundary conditions at all boundaries are required. Using the momentum equations at the boundaries, Neumann boundary conditions are obtained:

along constant-\(\eta\) lines (AB and FEDC)
\[
\{J(-\beta_\zeta + \gamma_\eta)\}_\xi = \frac{\partial}{\partial t} \left(\frac{V}{J}\right)_\eta + \frac{1}{Re} \omega_\zeta \tag{10a}\]

along constant-\(\xi\) lines (AF and BC)
\[
\{J(\alpha_\xi - \beta_\eta)\}_\eta = \frac{\partial}{\partial t} \left(\frac{U}{J}\right)_\xi + \frac{1}{Re} \omega_\eta \tag{10b}\]

where
\[
\omega = J\left((ux_\eta + vy_\eta)\xi - (ux_\xi + vy_\xi)\eta\right). \tag{10c}\]

The diffusion terms in the boundary conditions, Eqs. (10a, b), are written in terms of the vorticity following the method of reference [14, 18]. Solutions for Eq. (8) with the Neumann boundary conditions, Eqs. (10a, b), exist only if the following compatibility condition is satisfied.

**Compatibility Condition**

The compatibility condition results from Green's theorem which relates the source term of the Poisson equation, Eq. (8), and the Neumann boundary conditions as follows
\[
\iint_\Omega \sigma dA = \oint_C \frac{\partial p}{\partial n} ds \tag{11}\]

where \(C\) is the contour enclosing the area of the solution domain \(\Omega\).

It is shown in [14] that the compatibility condition, Eq. (11), is not automatically satisfied on Cartesian non-staggered grids. A similar conclusion can be drawn in generalized curvilinear coordinates. In order to satisfy the compatibility condition, Eq. (11), the method of [14, 18] is extended here to curvilinear coordinates. It is important to stress here that the cross-derivative terms of the pressure and velocity in the pressure Poisson equation should be handled as shown in Eq. (8).

**NUMERICAL SOLUTIONS**

In this section, the finite difference equations approximating the momentum and pressure equations are obtained.
The Momentum Equations

The velocity components $u$ and $v$ are calculated from the solution of Eqs.(5) and (6) by marching in time. An explicit central finite difference approximation for the convection and diffusion terms in Eqs.(5) and (6) give second-order accurate solutions. However, at high Reynolds number, the finite difference equations approximating Eqs.(5) and (6) lose diagonal dominance and display non-physical oscillation [1]. In the present study, a new scheme which is stable at all Reynolds numbers is used for the solution of Eqs.(5) and (6). The method is second-order accurate in space. The scheme is briefly outlined here and for more details see [20].

Second-Order Accurate Scheme for the Convection-Diffusion Equation. The method is based on approximating the convection terms by using forward and backward derivatives. For example,

$$u_f = \frac{u}{2} \left\{ (f_\xi)_F + (f_\xi)_B \right\},$$

(12)

where the subscripts $F$ and $B$ refer to forward and backward, respectively. Depending on the direction of the flow, these derivatives are approximated as follows:

If $u > 0$,

$$u_f = \frac{u}{2} \left\{ (f_\xi)_F - (f_\xi)_B \right\},$$

(13a)

The forward derivative is approximated as the average of at the locations $y + \Delta y$ and $y - \Delta y$. That is,

$$\frac{1}{\Delta \xi} \left( f_{i+1,j} - f_{i,j} \right)$$

(13b)

Similarly, for $u < 0$,

$$u_f = \frac{u}{2} \left\{ (f_\xi)_F - (f_\xi)_B \right\}$$

(14a)

$$u_f = \frac{u}{2} \left\{ (f_\xi)_B - (f_\xi)_F \right\}$$

(14b)

Although each of the above approximations is first-order accurate, their summation in Eq.(12) is second-order accurate. Therefore, for $u > 0$,

$$u_f = \frac{u}{2} \left\{ (f_\xi)_F - (f_\xi)_B \right\}$$

(15a)

for $u < 0$,

$$u_f = \frac{u}{2} \left\{ (f_\xi)_B - (f_\xi)_F \right\}$$

(15b)

Similar expressions can be obtained for derivatives in the $\eta$-direction.

The diffusion terms in Eqs. (5) and (6) are approximated by use of central second-order finite difference expressions. Upon substitution of the above difference expressions in Eqs.(5) and (6), one obtains

$$u_{i+1,j}^n = u_{i,j}^n - \Delta t \left( \frac{J^2 \tau}{Re} \right)_{i,j} \left( \delta_x u \right)_{i,j} - \Delta t \left( \frac{J^2 \theta}{Re} \right)_{i,j} \left( \delta_\eta u \right)_{i,j}$$

$$- J_{i,j}^{\Delta t} \left\{ \left( y_n p_{i+1,j} \right)_{i+1,j} - \left( y_n p_{i,j} \right)_{i,j} \right\}$$

$$+ \frac{J^2 \tau}{Re} \left( \alpha \left( u_{i+1,j} - u_{i,j} \right) - \beta \left( u_{i+1,j} + u_{i,j} \right) - \gamma \left( u_{i,j} + u_{i+1,j} \right) \right)$$

(16a)

and

$$v_{i,j}^n = v_{i,j}^n - \Delta t \left( \frac{J^2 \tau}{Re} \right)_{i,j} \left( \delta_\xi v \right)_{i,j} - \Delta t \left( \frac{J^2 \theta}{Re} \right)_{i,j} \left( \delta_\eta v \right)_{i,j}$$

$$+ J_{i,j}^{\Delta t} \left\{ \left( x_n p_{i+1,j} \right)_{i+1,j} - \left( x_n p_{i,j} \right)_{i,j} \right\}$$

(16b)

where $\delta_\xi$ and $\delta_\eta$ are the differencing operators using the new second-order accurate scheme in the $\xi$ and $\eta$-directions, respectively. The right hand sides of Eqs.(16a, b) are evaluated at the time level $n - 1$.

Finite Difference Equations for the Pressure Equation

Using the method of [14, 18, 19] and referring to Fig.2, Eq.(8) is approximated by

$$a_{i,j} \left( P_{i,j+1} - P_{i,j-1} \right) - a_{i+1,j} \left( P_{i+1,j} - P_{i,j} \right) - a_{i,j+1} \left( P_{i,j+1} - P_{i,j} \right) + a_{i,j-1} \left( P_{i,j-1} - P_{i,j} \right) = \frac{J^2 \Delta t}{Re} \left( \alpha \left( u_{i+1,j} - u_{i,j} \right) - \beta \left( u_{i+1,j} + u_{i,j} \right) - \gamma \left( u_{i,j} + u_{i+1,j} \right) \right)$$

(17a)

$$+ \frac{J^2 \tau}{Re} \left( \alpha \left( u_{i+1,j} - u_{i,j} \right) - \beta \left( u_{i+1,j} + u_{i,j} \right) - \gamma \left( u_{i,j} + u_{i+1,j} \right) \right)$$

(17b)

$$+ \frac{J^2 \tau}{Re} \left( \alpha \left( v_{i,j+1} - v_{i,j} \right) - \beta \left( v_{i,j+1} + v_{i,j} \right) - \gamma \left( v_{i,j} + v_{i,j+1} \right) \right)$$

(17c)

$$+ \frac{J^2 \tau}{Re} \left( \alpha \left( w_{i,j} - w_{i,j} \right) - \beta \left( w_{i,j} + w_{i,j} \right) - \gamma \left( w_{i,j} + w_{i,j} \right) \right)$$

(17d)

where the terms in the left hand side are evaluated at the time level $n$, with

$$\left( \alpha \right)_{i,j} = \alpha \left( p_{i+1,j} - p_{i,j} \right)$$

(18a)

$$\left( \beta \right)_{i,j} = \frac{1}{4} \left( \beta_\xi \left( p_{i+1,j+1} + p_{i,j+1} - p_{i+1,j-1} - p_{i,j-1} \right) \right)$$

(18b)

Similar expressions at $w, n$, and $s$ are obtained using the same procedure. Therefore, the discretized pressure equation is

$$a_{i-1,j-1} \left( P_{i-1,j-1} - P_{i,j-1} \right) + a_{i-1,j} \left( P_{i-1,j} - P_{i,j} \right) + a_{i,j-1} \left( P_{i,j-1} - P_{i,j} \right)$$

$$+ a_{i,j} \left( P_{i,j} - P_{i+1,j} \right) + a_{i+1,j-1} \left( P_{i+1,j-1} - P_{i,j-1} \right)$$

$$+ a_{i+1,j} \left( P_{i+1,j} - P_{i,j} \right) = \frac{J^2 \Delta t}{Re} \left( \alpha \left( u_{i+1,j} - u_{i,j} \right) - \beta \left( u_{i+1,j} + u_{i,j} \right) - \gamma \left( u_{i,j} + u_{i+1,j} \right) \right)$$

(19)

where the coefficients $a_{i-1,j-1}, \ldots, a_{i+1,j+1}$ are given by

$$a_{i,j} = \frac{J^2}{Re} \left( \frac{\Delta x}{\Delta y} \right)^2 \left( \alpha \left( u_{i,j} \right) - \beta \left( u_{i,j} \right) - \gamma \left( u_{i,j} \right) \right)$$

(20a)

$$a_{i-1,j} = \frac{J^2}{Re} \left( \frac{\Delta x}{\Delta y} \right)^2 \left( \alpha \left( u_{i-1,j} \right) - \beta \left( u_{i-1,j} \right) - \gamma \left( u_{i-1,j} \right) \right)$$

(20b)

$$a_{i,j-1} = \frac{J^2}{Re} \left( \frac{\Delta x}{\Delta y} \right)^2 \left( \alpha \left( u_{i,j-1} \right) - \beta \left( u_{i,j-1} \right) - \gamma \left( u_{i,j-1} \right) \right)$$

(20c)

$$a_{i+1,j} = \frac{J^2}{Re} \left( \frac{\Delta x}{\Delta y} \right)^2 \left( \alpha \left( u_{i+1,j} \right) - \beta \left( u_{i+1,j} \right) - \gamma \left( u_{i+1,j} \right) \right)$$

(20d)

$$a_{i,j+1} = \frac{J^2}{Re} \left( \frac{\Delta x}{\Delta y} \right)^2 \left( \alpha \left( u_{i,j+1} \right) - \beta \left( u_{i,j+1} \right) - \gamma \left( u_{i,j+1} \right) \right)$$

(20e)

$$a_{i-1,j+1} = \frac{J^2}{Re} \left( \frac{\Delta x}{\Delta y} \right)^2 \left( \alpha \left( u_{i-1,j+1} \right) - \beta \left( u_{i-1,j+1} \right) - \gamma \left( u_{i-1,j+1} \right) \right)$$

(20f)

$$a_{i+1,j+1} = \frac{J^2}{Re} \left( \frac{\Delta x}{\Delta y} \right)^2 \left( \alpha \left( u_{i+1,j+1} \right) - \beta \left( u_{i+1,j+1} \right) - \gamma \left( u_{i+1,j+1} \right) \right)$$

(20g)
\[
a_{i-1,j-1} = -\frac{1}{4} \{ (J\beta)_{i-\frac{1}{2},j} + (J\beta)_{i,j-\frac{1}{2}} \}, \quad (20a)
\]
\[
a_{i-1,j} = (J\alpha)_{i-\frac{1}{2},j} + \frac{1}{4} \{ (J\beta)_{i,j+\frac{1}{2}} - (J\beta)_{i,j-\frac{1}{2}} \}, \quad (20b)
\]
\[
a_{i-1,j+1} = \frac{1}{4} \{ (J\beta)_{i-\frac{1}{2},j} + (J\beta)_{i,j+\frac{1}{2}} \}, \quad (20c)
\]
\[
a_{i,j-1} = (J\gamma)_{i,j-\frac{1}{2}} + \frac{1}{4} \{ (J\beta)_{i-\frac{1}{2},j} - (J\beta)_{i,j+\frac{1}{2}} \}, \quad (20d)
\]
\[
a_{i,j+1} = (J\alpha)_{i+\frac{1}{2},j} + \frac{1}{4} \{ (J\beta)_{i,j-\frac{1}{2}} - (J\beta)_{i,j+\frac{1}{2}} \}, \quad (20e)
\]
\[
a_{i,j-1} = (J\gamma)_{i,j+\frac{1}{2}} + \frac{1}{4} \{ (J\beta)_{i-\frac{1}{2},j} - (J\beta)_{i,j+\frac{1}{2}} \}, \quad (20f)
\]
\[
a_{i+1,j-1} = \frac{1}{4} \{ (J\beta)_{i+\frac{1}{2},j} + (J\beta)_{i,j-\frac{1}{2}} \}, \quad (20g)
\]
\[
a_{i+1,j+1} = (J\alpha)_{i+\frac{1}{2},j} + \frac{1}{4} \{ (J\beta)_{i,j-\frac{1}{2}} - (J\beta)_{i,j+\frac{1}{2}} \}, \quad (20h)
\]

The source term \( \sigma_{ij} \) is approximated as follows
\[
\sigma_{ij} = Q - Q_e + S_a - S_n \quad (21)
\]
where
\[
Q = y_T(Uu_x + Uv_u) - x_n(Uu_x Vv_n) \quad (22a)
\]
and
\[
S = y_T(Uu_x + Uv_u) - x_n(Uu_x Vv_n) \quad (22b)
\]
As suggested by Harlow and Welch [12], to cure the instability arising from the differencing of the unsteady term, \( \partial D/\partial t \), only \( D^{n-1} \) is retained in Eq.(19).

The Neumann boundary conditions, Eqs.(10a, b), are discretized at the half point from the boundaries. Following Abdallah's differencing procedure[14, 18], the boundary condition along the left constant-\( \xi \) line is discretized to
\[
(J\alpha)_{\frac{1}{2},j} (p_{2,j} - p_{1,j})^n - \frac{1}{4} (J\beta)_{\frac{1}{2},j} (p_{2,j+1} + p_{1,j+1} - p_{2,j-1} - p_{1,j-1})^n
\]
\[
= \frac{1}{2} (Q_{1,j} + Q_{2,j})^n + \frac{1}{4 Re} (\omega_{2,j+1} + \omega_{1,j+1} - \omega_{2,j-1} - \omega_{1,j-1})^n
\]
\[
- \frac{1}{2\Delta t} \left[ \left( \frac{U^n}{J} \right)_{1,j} + \left( \frac{U^n}{J} \right)_{2,j} - \left( \frac{U^{n-1}}{J} \right)_{1,j} - \left( \frac{U^{n-1}}{J} \right)_{2,j} \right], \quad (23)
\]
The differencing of the boundary conditions along the other three sides can be obtained using the same procedure. Note that at each time step the pressure equation, Eq.(19), with the boundary conditions, Eq.(24), are solved using the successive over-relaxation method. The successive over-relaxation factor used here was varied between 1.0 and 1.7 without significant effects on the convergence speed. For steady flow, the pressure equation is not iterated to full convergence at each time step. However, for unsteady flow solutions, the pressure equation must be iterated to full convergence in order to satisfy the continuity equation at each time step. In the present calculations, the pressure equation is solved for 10 iterations at each time step to produce smooth numerical solution for the pressure.

**COMPATIBILITY CONDITION**

The finite-differencing procedure given above satisfies the compatibility condition, Eq.(11), exactly. The summation of the right and left hand sides of Eqs.(21) and (24) is equivalent to the compatibility condition in discrete form [14]. It can be shown that
\[
\text{R.H.S.} = 0
\]
and
\[
\text{L.H.S.} = 0.
\]
Hence, the compatibility condition is numerically satisfied. It should be noted here that the viscous terms in the boundary conditions, Eq.(21), cancel. This is the main reason for writing the viscous terms in terms of the vorticity.

**ITERATIVE PROCEDURE**

Starting from arbitrary conditions, the iterative procedure for solving the governing equations, Eqs.(16a, b) and (19), for \( u, v, \) and \( p \) consists of three steps:
1. Knowing \( p^{n-1} \) (initially arbitrary), solve the momentum equations, Eqs.(16a, b), for \( u^n \) and \( v^n \), respectively.
2. Knowing the velocity field, solve the Poisson equation, Eq.(21), for \( p^n \).
3. Repeat steps 1 and 2 until convergence is reached.

**RESULTS AND DISCUSSIONS**

Solutions for the governing equations, Eqs.(16a, b) and (19), are obtained for both inviscid and viscous flows in the blade passage shown in Fig.1. Because of symmetry, only one half of the solution domain is considered for numerical solutions.

Fig.3 shows the computational grid used for the inviscid case. The inviscid grid has 77 x 41 grid points with an initial spacing next to the body of 0.0079 chord. The viscous grid shown in Fig.4 has 121 x 57 grid points with an initial spacing of 0.0051 chord.

![Fig. 3 Computational grid for inviscid flow.](image-url)

![Fig. 4 Computational grid for viscous flow.](image-url)
Convergence histories for the inviscid case is shown in Fig.5, which plots the average of residues for \( u, v, \) and \( p \) versus iteration. The particular explicit method used here for solving the momentum equations is limited by a small time step which is controlled by the small grid spacing near the body. It requires about 5000 steps to converge all of the averages of residues to \( 10^{-7} \). A fine grid is used in the inviscid case because the upwind scheme used for solving the momentum equations is first-order accurate. The computational time used for solving this problem is about 330 minutes on VAX/VMS785.

Fig.6 compares the present results with the results of [8] which were obtained by use of the artificial compressibility method. These results were obtained using \( 121 \times 59 \) grid points with a second-order accurate method. The presented results show that the two methods, which are completely different in the way they calculate the pressure, are in good agreement.

The results for the viscous cases are obtained at \( Re = 100 \) for the same blade passage of Fig.1. The convergence curves which are shown in Fig.7 are typical for the second-order accurate scheme which is used for the solution of the momentum equations. Fig.8 shows the comparison of the present results for viscous flows.
CONCLUSION

In the present study, Abdallah's consistent finite difference method (CFDM) [14, 18, 19] for the solution of the pressure equation of incompressible flows has been extended to generalized curvilinear coordinates. Care must be exercised when the method is extended to generalized curvilinear coordinates because of the presence of cross-derivative terms in the pressure Poisson equation. The method satisfies the compatibility condition of the pressure Poisson equation on non-staggered grids, leading to convergent solutions. The CFDM is applied to calculating both inviscid and viscous flows in a blade passage. The viscous momentum equations are solved using a second-order accurate scheme which is developed by the authors and will be reported in a separate study [20]. The computed results show that the CFDM converges and compares well with the artificial compressibility approach. The pressure Poisson equation approach is valid for both steady and unsteady flow solutions.

REFERENCES


