Numerical Solution to Transonic Potential Equations on $S_2$ Stream Surface in a Turbomachine

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Abstract

A non-isentropic potential method on a $S_2$ stream surface has been developed for the design and analysis of transonic compressors with shocks, in which the entropy increase across a shock may be directly calculated from the momentum equations in the divergence form. The numerical results show that the non-isentropic shock is weaker and placed one or two meches further upstream compared to the classical potential calculation, and is in good agreement with the experimental data.

Introduction

In recent years, the full-potential formulation has been highly developed and widely used in both external and internal transonic flow problems (e.g. Hafez et al., 1978, Lu and Wu, 1985). To develop this approach further and to improve its accuracy of calculations in transonic regime it is necessary to get rid of the isentropic assumption. In this respect a non-isentropic potential formulation was suggested first by Klopfer and Nixon (1983) and it was pointed out that its results are very close to the Euler solutions. Then a Poisson equation for the pressure was derived from the momentum equations and was solved numerically. (Hafez et al. 1984)

Recently, a method to calculate the entropy increase across a shock is investigated by the present authors (Xu et al. 1987) and will be applied to the throughflow calculations of transonic compressors. In this method instead of the pressure, the entropy variation is determined from the momentum equations in divergence form and the whole calculation is simplified to the iterations between the potential, the density and the entropy. It makes the results of the throughflow calculation closer to the solutions of Euler equations. It is expected that this non-isentropic potential method on the $S_2$ stream surface will find some applications in analysing the internal flow field of the transonic compressors.

Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$f$</td>
<td>force between $S_2$ stream surfaces</td>
</tr>
<tr>
<td>$g$</td>
<td>mass flow rate</td>
</tr>
<tr>
<td>$g_{ij}$</td>
<td>determinant of the metric tensor</td>
</tr>
<tr>
<td>$g'$</td>
<td>metric tensor elements of three-dimensional coordinate system</td>
</tr>
<tr>
<td>$h$</td>
<td>enthalpy of gas per unit mass</td>
</tr>
<tr>
<td>$I$</td>
<td>relative stagnation rotality of gas</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure of gas</td>
</tr>
<tr>
<td>$R$</td>
<td>gas constant</td>
</tr>
<tr>
<td>$r$</td>
<td>radius, radial coordinate</td>
</tr>
<tr>
<td>$s$</td>
<td>entropy of gas per unit mass</td>
</tr>
<tr>
<td>$T$</td>
<td>change of entropy of gas per unit mass</td>
</tr>
<tr>
<td>$T$</td>
<td>absolute temperature of gas</td>
</tr>
<tr>
<td>$V$</td>
<td>absolute velocity of gas</td>
</tr>
<tr>
<td>$W$</td>
<td>relative velocity of gas</td>
</tr>
<tr>
<td>$w_i$</td>
<td>component of $\mathbf{W}$</td>
</tr>
<tr>
<td>$x_i$</td>
<td>nonorthogonal curvilinear coordinates</td>
</tr>
<tr>
<td>$k$</td>
<td>ratio of specific heats of gas</td>
</tr>
<tr>
<td>$v$</td>
<td>artificial viscous coefficient</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of gas</td>
</tr>
<tr>
<td>$\rho_0$</td>
<td>artificial density</td>
</tr>
<tr>
<td>$\phi$</td>
<td>potential function</td>
</tr>
<tr>
<td>$\theta$</td>
<td>relative tangential coordinates</td>
</tr>
<tr>
<td>$\omega$</td>
<td>angular velocity of rotor</td>
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Superscript

$i$ = contravariant component of vector

Subscripts

$0$ = stagnation state

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\[ I \] = reference state where \( h = 1 \)
\[ i \] = covariant component of vector
\[ \theta \] = component of vector along absolute tangential coordinate of cylindrical coordinate
\[ \phi \] = component of vector along relative tangential coordinate of cylindrical coordinate

**POTENTIAL EQUATION ON A GIVEN \( S_2 \) STREAM SURFACE AND ITS NUMERICAL METHOD**

When a irrotational flow in the absolute motion is considered, the basic equations for the steady relative flow of a non-viscous gas along a given \( S_2 \) stream surface may be written as

\[
\nabla \cdot (B \rho \vec{W}) = 0
\]
\[
\nabla \times \vec{V} = 0
\]
\[
dI/dt = 0
\]
\[
Tds = dh - \frac{1}{2} dP
\]

where \( I \) is the relative stagnation rothalpy (Wu, 1952).

From Eq.(2) a potential function \( \phi \) may be introduced:

\[
V\vec{e} = V
\]

Expressed with respect to non-orthogonal curvilinear coordinates (Wu, 1976), in which \( x^1 \) and \( x^2 \) lie in the meridional plane and \( x^3 = \phi \), the continuity equation (1) takes the form:

\[
(B \rho \vec{W})_{x^1} \sin \theta_2 + (B \rho \vec{W})_{x^2} \sin \theta_2 = 0
\]

Substituting Eq.(5) into Eq.(1a) gives the potential equation in a divergent form (Lu and Wu, 1985):

\[
[D_1 \rho (A_1 \vec{e}_{x^1} + A_2 \vec{e}_{x^2} + A_3) + D_2 \rho (B_1 \vec{e}_{x^1} + B_2 \vec{e}_{x^2} + B_3)] = 0
\]

with

\[
D_1 = \rho (g''_{x^1} + g''_{x^2}, A_1 = g''_{x^1}, A_2 = g''_{x^2}, B_2 = g''_{x^2})
\]

\[
A_3 = V = \rho (g''_{x^1} + g''_{x^2}, B_3 = \rho (g''_{x^1} + g''_{x^2})
\]

In order that the solution of Eq.(6) be stable and convergent in the transonic regime the artificial density (Hafez, 1978) in the following form is used at supersonic grid points:

\[
\rho_i = \rho_k - \left( \frac{V_i - V_{i-1}}{V_i - V_{i+1}} \right) \text{ for } W_{i,j} > 0
\]
\[
\rho_i = \rho_k - \left( \frac{V_i - V_{i+1}}{V_i - V_{i-1}} \right) \text{ for } W_{i,j} < 0
\]

where

\[
V_i = \begin{cases} \text{Max} \left[ 0, \frac{C (M_{i,j+1} - 1) }{M_{i,j+1}} \right] & \text{for } W_{i,j} > 0 \\ \text{Max} \left[ 0, \frac{C (M_{i,j-1} - 1) }{M_{i,j-1}} \right] & \text{for } W_{i,j} < 0 \end{cases}
\]

and \( C \) is a parameter governing the strength of the artificial viscosity terms. The numerical experiments show that the range of \( C \) is from 0.5 to 1.5.

As the boundary conditions the stagnation temperature, the stagnation pressure, the relative flow angle and the mass flow rate are given and the radial velocity is assumed to be zero at the inlet boundary, so the radial distribution of the inlet axial velocity can be obtained by the radial equilibrium condition, while the circumferential velocity or the flow angle is specified and the radial velocity is considered to be zero at the exit boundary, and the variation of axial velocity can be determined as in the inlet case. On the solid walls, the flow tangency requires \( w^2 = 0 \).

For \( V_{r,r} \) appeared in the coefficients \( A_3 \) and \( B_3 \) of Eq.(6) the following relation is used:

\[
\nu_{i,j} = \nu_{i,j} + 1 \left( \frac{1}{2} \left( w_{i,j} + w_{i,j}^* \right) \right)
\]

**NON-ISENTROPIC POTENTIAL FORMULATION AND ENTROPY EQUATION**

In the non-isentropic potential formulation the density in Eq.(6) is related to the change of entropy:

\[
\rho = \rho_0 \left( 1 - \frac{1}{2} \left( w_{i,j} + w_{i,j}^* \right) \right)^{\frac{1}{2} \left( w_{i,j} + w_{i,j}^* \right)}
\]

where \( \nu \) denotes the entropy variation, \( s = s_0 \).

To calculate the entropy increase pass through a shock it is necessary to use the momentum equations in divergence form:

\[
\frac{d}{dt} \left( \rho \vec{V} \right) = \rho (\vec{F} - \vec{g} - \vec{E})
\]

when the equation of state is considered, the right-hand side of the momentum equations becomes

\[
\nabla P = \frac{\rho}{\rho - 1} \left( \rho \nabla T - \rho T \nabla s \right)
\]

It is seen that the entropy variation may be obtained from the momentum equations directly. In order to
ensure the accuracy and the reliability of the numerical results, it is better to differentiate Eqs.(8a) and (8b) and to plus them to form a Poisson equation for $s$:

$$\frac{\partial}{\partial x}\left(\frac{1}{2} \rho u^2 \right) + \frac{\partial}{\partial y}\left(\frac{1}{2} \rho u v \right) + \frac{\partial}{\partial z}\left(\frac{1}{2} \rho u w \right) = \frac{d\rho}{dx} + \frac{d\rho}{dy} + \frac{d\rho}{dz}$$

where

$$C_1 = \left(\frac{u^2 - w^2}{2} + \frac{u^2}{2}\right)\left(\frac{d\rho}{dx}\right) - \frac{2}{3}w^2\rho S_{\text{eff}} \left(\frac{d\rho}{dx}\right) - \frac{2}{3}w^2\rho S_{\text{eff}} \left(\frac{d\rho}{dx}\right) - \left(\frac{w^2}{2}\right)\left(\frac{d\rho}{dx}\right)$$

$$C_2 = -2\left[w \left(\frac{du}{dx}\right) + \frac{u^2 - w^2}{2} + \frac{u^2}{2}\right] + \left(\frac{w^2}{2}\right)\left(\frac{d\rho}{dx}\right) - \frac{2}{3}w^2\rho S_{\text{eff}} \left(\frac{d\rho}{dx}\right)$$

It is noted that on the right-hand side of Eq.(9) all terms are in the divergence form. It is easy to see that the shocks computed from this non-isentropic potential formulation will meet the Rankine-Hugoniot condition approximately and its numerical solution will be very close to Euler solution. So this method is more rigorous in theory and very simple in computation and will extend the range of validity of the potential formulation.

The boundary conditions for Eq.(9) are composed of three parts. At the upstream boundary $s$ is regarded as zero and at the downstream boundary $s$ is assumed to be constant along a streamline. The condition on the solid walls may be derived by means of integrating Eq.(9) by parts.

Both Eqs.(6) and (9) are solved by the line relaxation method in this paper.

SOLUTION PROCEDURE AND NUMERICAL RESULTS

As it has been shown that the basic equations in the present non-isentropic throughflow calculations are Eqs.(6), (7) and (9). The whole calculation procedure then consists of the iterations between these equations. At the beginning of calculation, an initial distribution of $s$ may be computed based on an assumed distribution of the velocity. Density $\rho$ is calculated according to Eq.(7), taken $s$ to be zero. Then Eq.(6) is solved for a new distribution of $s$ and Eq.(9) is solved for a field of $s$ successively. By use of of Eq.(7) an improved density $\rho$ may be updated and it contains the influence of entropy at this moment. Starting from the distributions of the density and other gas variables, the next cycle of iterations begins and this procedure will continue until the convergence criterion is satisfied. In this study the criterion is chosen as the maximum relative change in the relative velocities at all grid points in the throughflow field to be less than a prescribed precision, say $0.4 \times 10^{-3}$. It is expected that owing to the introduction of entropy, the convergence of the present non-isentropic potential calculation will be a little bit better and the increase in whole CPU time is a little compared to the classical potential calculation.

Several examples of transonic throughflow calculations have been carried out to test the effectiveness of this method. Here some numerical results of a DFVLR transonic rotor with experimental data and NASA Rotor 6 are cited.

The geometries of the DFVLR rotor are given by McDonald et al. (1980). The flowpath and the computational domain are shown in Fig.1. The inlet conditions in the calculation are: $G = 17.3$ kg/s, $\omega = 20260$ r/min, $T_0 = 288.2$ K, $P_0 = 101600$ N/m$^2$. In the exit the values of $V_g$ are specified according the measured isentropic efficiency and the stagnation pressure. A 63 x 11 grid is used in the calculation and 31 meshes are located within the rotor. The thickness of the $s$ streamfilament is simply assumed to be proportional to the circumferential arc length of the flow channel and the sudden turning of the given $s$ stream surface across the shock is neglected (Xu et al., 1982). The calculation does not account for the effects of viscosity, only a 2.5 per cent annulus blockage being considered.

![Fig.1 Flow path and computational domain](image-url)
It is interesting to note that the location of the non-isentropic shock on the S2 stream surface is one or two meshes upstream compared to the isentropic one which is about 6-9 per cent chord behind the location obtained in the test. Moreover, the maximum Mach numbers in front of the shock in the non-isentropic calculation are lower than those in the isentropic calculation and the Mach numbers ahead of the shock are almost the same. So the non-isentropic shock is weaker than the isentropic one.

From the chordwise distribution of the relative Mach numbers (Fig.3) it is seen that corresponding to Fig.2, Mach numbers in the non-isentropic calculation drop suddenly at about 3-4 per cent chord behind that in the isentropic calculation and is hence closer to the experimental data. As Mach number before the shock increases, the influence of the entropy variation on the shock location is also enhanced. Even for the flow with a inlet Mach number less than 1.3, Mach number behind the shock may be greater than 1.3 and the non-isentropic potential calculation is desirable.

The chordwise variation of the calculated angular momentum, $V_{gr}$, is given in Fig.4. It shows that when a shock occurs the angular momentum after the shock may be less than that in the inlet and the gas is over-turned. But the decrease of $V_{gr}$ and the overturning of the gas in the non-isentropic potential model are reduced in comparison with the isentropic calculation. With the increasing of Mach number the entropy increase caused by the shock raises and the differences between these two calculations become considerable.

The flowpath and the calculation results of NASA Rotor 6 are given in Fig.5. The following inlet conditions are used: $G = 29.76$ kg/s, $\omega = 16002$ r/min, $T_I = 288.2$ k, $P_I = 101600$ N/m . The exit condition is the measured circumferential velocity component, $V_0$. Compared with the only test result (Kovich and Reid, 1973), it is found that the computed relative Mach number at the leading and trailing edges are quite accurate. It seems that as the inlet Mach number increases, the meridional projection of the shock in this rotor is closer to the leading edge than that in the DFVLR rotor. This position of the non-isentropic shock is one mesh upstream of that in the isentropic calculation which has not drawn in Fig.5 to avoid excessive complication of the figure.

Fig.2 Meridional projection of computed Mach number contours along mean S2 stream surface

Fig.3 Chordwise distribution of Mach numbers on mean S2 stream surface

Fig.4 Chordwise distribution of $V_{gr}$ on mean S2 stream surface

Fig.5 Meridional projection of computed Mach number contours along mean S2 stream surface for NASA Rotor 6
On these two examples and other numerical results it comes a conclusion that for the throughflow calculations of a transonic compressor the present non-isentropic potential formulation provides a simple and effective method.

From the calculations it is found that the iteration number of the non-isentropic potential is a little bit less than the isentropic one to meet the same convergence precision due to the introduction of the entropy and the overall computing time increases only slightly.

CONCLUDING REMARKS

On the basis of calculating the entropy increase across the shock from the momentum equations in divergence form, a non-isentropic potential throughflow calculation method has been developed in this paper. The present method is more strict in theory and its solution is close to Euler solution. Furthermore, it keeps the superiority in computation of the isentropic potential formulation and has a little bit better convergence than the latter. Because of the capability in accurately treating the transonic flows with Mach numbers greater than 1.3 and the effectiveness in computation, it is expected that this method will become a useful tool in the design and analysis of transonic compressors with shocks.

The computer code developed was used to calculate the transonic throughflow of the DFVLR transonic rotor and NASA Rotor 6. The results show that the non-isentropic shocks are located one or two meshes further upstream compared to the isentropic solution. The shocks are weaker and the jumps of the gas variables across the shocks are also reduced. All these are in good agreement with the experimental results.

REFERENCES