TURBULENT SPOT CHARACTERISTICS IN BOUNDARY LAYERS SUBJECTED TO STREAMWISE PRESSURE GRADIENT

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ABSTRACT

The development of linear perturbations within laminar boundary layers subjected to a full range of adverse and favourable pressure gradients has been predicted numerically. Comparison of the predictions with published flow visualisation results enabled identification of the perturbation regions associated with the turbulent spot, trailing vortex streaks and oblique waves along the spot flanks. The effect of streamwise pressure gradient on spot characteristics (spot leading and trailing edge velocity, calming region trailing edge velocity and spot spreading half angle) was determined and was found to be consistent with published experimental data. The new data extends the range to severe adverse pressure gradient (separation) and to stronger favourable gradients. Correlations for the spot characteristics are provided.

NOMENCLATURE

- $p'$: Fluctuating pressure
- $Re_T = \frac{U|\delta|}{v}$: Boundary layer Reynolds number
- $t$: Time
- $T$: Dimensionless time
- $u$: Time mean velocity in streamwise direction
- $U$: Freestream time mean velocity
- $U_c$: Calmed region trailing edge velocity
- $U_{ce}$: Spot leading edge velocity
- $U_{te}$: Spot trailing edge velocity
- $U'$, $V'$, $W'$: Fluctuating velocities in $x$, $y$ and $z$ directions
- $V^*$: Amplitude of initiating pulse
- $x$, $y$, $z$: Streamwise, wall normal and spanwise coordinate
- $X$, $Y$, $Z$: Dimensionless coordinates
- $\alpha$: Spot spreading half angle
- $\beta$: Boundary layer thickness
- $\theta$: Boundary layer momentum thickness
- $\lambda = \frac{\delta^2}{U} \frac{dU}{dx}$: Pohlhausen pressure gradient parameter
- $\lambda_w = \frac{\delta^2}{U} \frac{dU}{dx}$: Pressure gradient parameter
- $v$: Kinematic viscosity
- $p$: Fluid density
- $c$: Cosine coefficient
- $i$: Fourier term ($x$ direction)
- $k$: Fourier term ($z$ direction)
- $\max$: Computational domain limit
- $p$: Dimensionless pulse width
- $s$: Sin coefficient

INTRODUCTION

The prediction of transition on gas turbine blades is achieved traditionally through the use of empirical correlations (e.g., Abu Ghannam and Shaw 1980). These correlations are generally sufficiently accurate in predicting start of transition, but less reliable in predicting transition length as no allowance is made for the effect of varying streamwise pressure gradient. More recent models (Solomon et al. (1995) and Johnson and Ercan (1996, 1997)) have included the effect of pressure gradient on the turbulent spot growth rate, which has resulted in more accurate transition length predictions. However, these models are reliant on the rather sparse experimental data on turbulent spot characteristics gathered from the literature by Gostelow et al. (1995). This data was also taken in a wide range of wind tunnels at different Reynolds numbers and using a variety of techniques to determine the extremities of the spots. For these reasons the correlations compiled by Gostelow et al. are subject to a large uncertainty, particularly where the growth rates are highest approaching laminar separation.
OBJECTIVE
The objective of the current work is to provide improved spot characteristic data which can be used to reduce the uncertainty in current spot characteristic correlations. Experiments to obtain this type of data over a wide range of streamwise pressure gradients are very time consuming and particularly difficult approaching separation. For these reasons, a numerical approach has been adopted here.

EQUATIONS OF MOTION
In the current work, the unsteady flow is assumed to be a small linear perturbation to the time mean flow. For this reason only the primary linear instabilities within the flow are determined rather than full breakdown to turbulence. Nevertheless the characteristics of the linearly disturbed region are very similar to those of the turbulent spot as shown previously for Poiseuille flow by Li and Widnall (1989).

The time mean flow is considered to be inviscid and parallel and to have a Pohlhausen laminar profile. Thus

\[
\frac{u}{U} = 2 \left( \frac{Y}{\delta} \right) - 2 \left( \frac{Y}{\delta} \right)^3 + \left( \frac{Y}{\delta} \right)^5
\]

(1)

\[
\frac{v}{V} = \frac{1}{6} \left( \frac{Y}{\delta} \right) - 3 \left( \frac{Y}{\delta} \right)^2 + \frac{1}{3} \left( \frac{Y}{\delta} \right)^3 - \left( \frac{X}{\delta} \right)^4
\]

(2)

The spot is assumed to be a small perturbation to this mean flow, but is considered to be fully three-dimensional and viscid. The unsteady momentum equations are

\[
u_i' + \frac{1}{\rho} p_i' - u_i' + u_i v_i' - v \nabla u_i' = 0
\]

(3)

\[
u_i' + \frac{1}{\rho} p_i' + u v_i' - v \nabla v_i' = 0
\]

(4)

\[
u_i' + \frac{1}{\rho} p_i' + u w_i' - v \nabla w_i' = 0
\]

(5)

Applying continuity

\[
u_x' + v_y' + w_z' = 0
\]

thus to the divergence of the momentum equations leads to

\[
\frac{1}{\rho} \nabla p' + 2u \nabla v = 0
\]

(6)

Now differentiating this equation with respect to y and eliminating \(p'\) using the Laplacian of equation (3) yields

\[
\nabla^2 v_i' - u \nabla^2 v_i' - u_{yy} v_i' - v \nabla^2 (\nabla^2 v) = 0
\]

(7)

This equation is now rewritten in terms of the dimensionless variables

\[
X = \frac{X}{\delta}, \quad Y = \frac{Y}{\delta}, \quad Z = \frac{Z}{\delta} \quad \text{and} \quad T = \frac{U}{U}
\]

Thus

\[
\frac{v_i'}{v_o} = \frac{-U}{U} \nabla^2 v_i + \frac{u_{yy} v_i}{U} \nabla v_i + \frac{1}{Re} \nabla^2 (\nabla^2 v) = 0
\]

(8)

Boundary Conditions
On the wall \(v = 0\) and also from continuity \(v' = 0\). If the spot has not propagated to reach any of the open boundaries, then these same Dirichlet and Neumann boundary conditions can be applied to the open boundaries as well.

Initial Conditions
The turbulent spot is initiated by introducing a pulse at a point on the wall. This is analogous to the manner in which spots are induced in wind tunnel tests (Gostelow et al. (1995)) where an electrical pulse excites a loudspeaker placed underneath a small hole in the wall on which a laminar boundary layer is developing. In the current work, the pulse has a profile

\[
\frac{v_i'}{v_o} = \frac{1}{4} \left( 1 + \cos \frac{2\pi X}{X_o} \right) \left( 1 + \cos \frac{2\pi Z}{Z_o} \right)
\]

(9)

for

\[
-\frac{X_o}{2} < X < \frac{X_o}{2} \quad \text{and} \quad -\frac{Z_o}{2} < Z < \frac{Z_o}{2}
\]

where \(X_o\) and \(Z_o\) are the pulse streamwise and spanwise widths. This is equivalent to

\[
v_i' = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\pi \left( 1 - i^2 X_o^2 \right) \left( 1 - k^2 Z_o^2 \right) i k} \sin \left( \pi n X \right) \sin \left( \pi k Z \right)
\]

(10)

SOLUTION PROCEDURE
As the coefficients of the \(v_i\) derivatives in equation (8) are functions of \(Y\) alone, it is convenient to use Fourier representations of \(\nabla^2 v\) in the \(X\) and \(Z\) directions. Thus,

\[
v_i' = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{v_i(i,k)}{X_{max}} \cos \frac{2\pi i X}{X_{max}} + \frac{v_i(i,k)}{Z_{max}} \sin \frac{2\pi i X}{Z_{max}} \cos \frac{2\pi k Z}{Z_{max}}
\]

(11)

where \(v_i(i,k)\) and \(v_i(i,k)\) are functions of \(Y\) alone.

Only Cos terms in \(Z\) are required as the flow can be assumed to...
be symmetric about \( Z = 0 \).

Equation (8) then becomes

\[
(\nabla^2 v)^{\lambda} = \frac{-2\pi i}{U \lambda_{\max}} v^2 \lambda_{\max} + \frac{u v}{U \lambda_{\max}} \left( \begin{array}{c} \frac{2\pi i}{U \lambda_{\max}} \lambda_{\max} + \frac{1}{\Re} \nabla^2 (\nabla v) \end{array} \right)
\]

and

\[
(\nabla^2 v)^{\lambda} = \frac{u v}{U \lambda_{\max}} \left( \begin{array}{c} \frac{2\pi i}{U \lambda_{\max}} \lambda_{\max} + \frac{1}{\Re} \nabla^2 (\nabla v) \end{array} \right)
\]

where

\[
\nabla^2 v = -\left( \frac{2\pi i}{U \lambda_{\max}} \right)^2 \left( \frac{2\pi k}{\lambda_{\max}} \right)^2 v^2 + v_{\text{new}}
\]

and

\[
\nabla^2 v = -\left( \frac{2\pi i}{U \lambda_{\max}} \right)^2 \left( \frac{2\pi k}{\lambda_{\max}} \right)^2 v^2 + v_{\text{new}}
\]

Equations (12) can be solved as ordinary differential equations in \( Y \) for each \( i,k \) Fourier term. An implicit finite difference scheme is used which is second order accurate in time and fourth order accurate in \( Y \).

In the current work 31 grid points were used in the \( Y \) direction with 41 and 21 Fourier terms in the \( X \) and \( Z \) directions, respectively. The computational domain measured 40 \( \times \) 3 \( \times \) 20 in the \( X, Y \) and \( Z \) directions. All the computations were performed for a \( Re = 4000 \) and an initiating pulse measuring \( X_i = 1 \) and \( Z_p = 1 \) of \( \Delta T = 0.05 \) duration.

A time step size of \( \Delta T = 0.05 \) was used.

RESULTS

Spot Development

Figure 1 shows the development of the spot within a zero pressure gradient (\( \lambda = 0 \)) mean boundary layer flow. The initial positive \( \sqrt{ } \) perturbation at the wall induces an upwash, which is seen as the leading positive perturbation region in the figure. A counteracting downwash region trails the upwash to replace the fluid removed. The overall flow pattern is therefore of an anticlockwise vortex. Up to \( T = 5 \), this vortex entrains additional fluid as it moves downstream at approximately 30% of freestream velocity. At \( T = 5 \), the core of the positive \( \sqrt{ } \) region is further from the wall (\( Y = 0.5 \)) than that of the negative \( \sqrt{ } \) region (\( Y = 0.25 \)). As the vortex grows it lifts away from the wall until the core of the positive \( \sqrt{ } \) region reaches the edge of the boundary layer at \( T = 30 \).

Figure 2 shows the plan view development at boundary layer mid-height. The perturbations convect most rapidly along the centreline which results in a 'teardrop' shape for the negative \( \sqrt{ } \) region and a 'heart' shape for the positive region at \( T = 5 \). This effect also distorts the vortex lines such that the initially spanwise vorticity develops into primarily streamwise vorticity. The overall pattern of the flow up to \( T = 10 \) is therefore of a 'kink' developing in a spanwise vortex line close to the wall (\( Y = 0.3 \)). This kink then develops into a hairpin vortex, the nose of which lifts from the surface as it lengthens.

At \( T = 10 \), Figure 2 shows that additional perturbation regions are developing ahead of the primary vortex. The leading positive \( \sqrt{ } \) region has separated from the nose of the main positive region as a result of the rapid stretching of the hairpin vortex once its leading edge is lifted away from the wall. The new negative \( \sqrt{ } \) regions appear to be induced by the vorticity in the legs of the hairpin vortex. At \( T = 20 \), the leading positive \( \sqrt{ } \) region has elongated substantially as its leading edge is convected rapidly in the upper part of the boundary layer. Necking of the elongated positive region would suggest that a further separation of the nose is imminent, but in fact this does not occur and the leading positive \( \sqrt{ } \) region grows in both spanwise and streamwise directions between \( T = 20 \) and 30.

If the current results at \( T = 30 \) are compared with the schematic results (Figure 3) from the flow visualisation work of Carlson et al. (1982), it is possible to identify the various flow features. The leading positive \( \sqrt{ } \) region, which has now adopted an approximately triangular shape, is coincident with the turbulent spot. The leading and trailing edge velocities of this region can be determined by comparing its position at \( T = 20 \) and \( T = 30 \) in Figure 2. The velocities of \( U_{\text{PE}}/U = 0.51 \) and \( U_{\text{TE}}/U = 0.89 \) found are consistent with other experimental results (Wygnanski, et al. (1982)). The negative \( \sqrt{ } \) regions which develop along the flanks of the spot do so

Figure 1. Spot development in the X-Y plane. \( \sqrt{v}/v \) contours. \( \lambda = 0 \). \( T = 1,2,5,10,20 \) and 50.
at an oblique angle which is consistent with the oblique waves observed in the flow visualisation study. The current work indicates that these regions move backwards and outwards relative to the spot to reach a position to the side of the spot’s tail. Each of these negative $\nu$ regions then induces a further positive $\nu$ region immediately in front of it. Thus, as the spot develops, alternate negative and positive $\nu$ regions are added on each side to the rear of the spot. These alternating $\nu$ regions constitute pairs of counter-rotating vortices which are seen as ‘streaks’ in the flow visualisation. The flow visualisation results distinguish the ‘oblique waves’ from these ‘streaks’ at the rear of the spot. This distinction is not clear in the current results, although a steep gradient in $\nu$ between adjacent $\nu$ regions would appear to constitute a ‘streak’, whereas an isolated region of negative $\nu$ would result in an ‘oblique wave’.

The ‘vortex streaks’ behind the spot remove low energy fluid from the near wall region and replace it with fresh fluid from the freestream. The result is that the boundary layer behind the spot is reduced in thickness and stabilised. The extent of this ‘calmed region’ is of importance in transition modelling in that new turbulent spots will not form within it.

The current results therefore show that linear perturbation theory is able to predict the geometrical features of the turbulent spot and the associated oblique waves and vortex streaks. Linear theory is not of course capable of predicting the development of turbulence, which is highly non-linear. However, the current results suggest that
turbulence develops within the leading positive $v'$ region, i.e., where there is an isolated region of flow away from the wall.

**Effect of Pressure Gradient**

Results from a full range of adverse and favourable pressure gradients at $T = 30$ are shown in Figure 4. For favourable pressure gradients ($\lambda > 0$) the development of the perturbations is similar to that for a zero pressure gradient, however, the development is accelerated with increasing $\lambda$ and the spanwise extent of the perturbations is much reduced. The strength of the perturbations in the spot region are also suppressed, which is indicative of the higher turbulence levels required to induce transition in a favourable pressure gradient.

For adverse pressure gradients ($\lambda < 0$), the flow pattern develops differently. The nose of the leading $v'$ region is no longer elongated and hence does not separate from the main region. New regions of $v'$ fluctuation are induced in front of the existing regions. The flow therefore consists of oblique wave packets. It would seem significant that this change in flow pattern with $\lambda$ occurs when the angle of the oblique waves becomes greater than 45°. The oblique waves lead to vortices, but when these are at an angle of less than 45° they will lead to predominantly spanwise ($v'w'$) mixing, whereas when their angle is greater than 45° they will lead to predominantly streamwise ($uvw'$) mixing.
Identification of Spot Characteristics

The spot and calming region were determined using a threshold value equal to the lowest contour value shown in Figure 4, i.e.,

$$\frac{v'}{v_o} = 10^{-4}$$

for the strong adverse pressure gradients where the perturbation levels are much higher. The effect of threshold level on the overall results was generally found to be small, although some influence was observed for the leading edge velocity. The leading edge was determined as the most forward point on the $z = 0$ plane. This point was between $Y = 0.3$ and 1. The trailing edge was taken as the rear most centreline point at $Y = 0.3$ on the most leading positive $\nabla$ region which could be identified at both $T = 20$ and $T = 30$. The spreading angle was determined from the spanwise extent of this region. The calming region results through the action of the vortex streaks. The trailing edge of this region was therefore taken as the rearmost point of the trailing positive $\nabla$ region at $Y = 0.3$.

Figure 5 shows the current values for the spot trailing edge and leading edge velocities and the calming region trailing edge velocity. Experimental data, over a modest pressure gradient range, collated from the literature by Gostelow et al. (1995) is also included. The current data is consistent with the experimental data and also extends the pressure gradient range from separation ($\lambda = -12$, $\lambda_a = -0.157$) to strong favourable pressure gradients ($\lambda = 12$, $\lambda_a = 0.095$).

Johnson and Ercan (1996) provided empirical correlations for the spot velocities based on the velocities existing in the Pohlhausen laminar boundary layer at heights $Y = 0.27$ and $Y = 0.57$ viz

\[ \text{(Equation 14)} \]
\[ \text{(Equation 15)} \]
\[ \text{(Equation 16)} \]

Figure 4. Effect of pressure gradient on spot development. $v'/v_o$ contours. $Y=0.3$. $T=30$. $\lambda=-12,-9,-6,-3,0,3,6,9$ and 12.

Figure 5. Spot leading and trailing edge velocities and calmed region trailing edge velocity.
The spot propagation parameter \( \sigma \) can be calculated as

\[
\sigma = \left( \frac{U}{U_{TB}} - \frac{U}{U_{LB}} \right) \tan \alpha
\]

if the spots are assumed to be triangular in shape. Figure 7 shows the \( \sigma \) values together with published experimental values and the correlations of Gostelow et al. (1995) and Johnson and Ercan (1996). Again the values are slightly higher than observed experimentally.

CONCLUSIONS

1. Linear perturbation theory is capable of predicting the geometrical development of a turbulent spot in a boundary layer subjected to a streamwise pressure gradient. The major features within the flow, namely the spot, the oblique waves and the trailing vortex streaks, can be identified.
2. The nature of the spot development alters for strong adverse pressure gradients where \( \lambda < -6 \). For \( \lambda > -6 \), the spot develops as a leading triangular positive \( V \) region. For \( \lambda < -6 \) the spot has the appearance of a series of oblique waves.

3. The present results indicate that a spot spreading half angle approaches, but does not reach, 90° at separation. This suggests that transition lengths for boundary layers close to separation will be extremely short.

REFERENCES


