A SIMPLE METHOD FOR SOLVING THREE-DIMENSIONAL INVERSE PROBLEMS OF TURBOMACHINE FLOW AND THE ANNULAR CONSTRAINT CONDITION

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ABSTRACT

In ASME paper 50-A-79, it was suggested that the three-dimensional inverse (design) problem of turbomachine flow may be solved approximately by a Taylor series expansion in the circumferential direction based on the known flow variation over an S2 stream surface in the mid-channel of the blade passage. This idea has been realized recently.

A new coordinate transformation has been developed. The coordinate surfaces are coincident with the S2 stream surfaces. This transformation leads to a new method to calculate the aerodynamic equations of 3-D flow. By the use of this transformation, a high order expansion is realized to determine the shape of the blade surfaces from the fluid state on the S2m stream surface directly.

Computation in this manner soon leads to the discovery that theoretically the distribution of flow parameter usually \( V_{\varphi} \) on S2m prescribed by the designer should satisfy a constraint condition, which guarantees that the S1 stream surfaces along the hub and shroud obtained from circumferential extension of the S2m surface are surfaces of revolution. An approximate method is suggested to meet this condition.

SYMBOLS

\( a_{ij} \quad \text{Basic metric tensor} \)

\( a \quad \text{Jacobian composed of } a_{ij} \)

\( B \quad \text{integrating factor in the continuity equation for S2 stream surfaces} \)

\( C \quad \text{nonzero term on the right-hand side of the continuity equation for S2 stream surface} \)

\( D_1, E_1, \ldots \quad \text{values of stream function's on iso-space of S1 family} \)

\( D_2, E_2, \ldots \quad \text{values of stream function's on iso-space of S2 family} \)

\( h \quad \text{enthalpy per unit mass of fluid} \)

\( I \quad \text{relative stagnation enthalpy, } (h-U^2/2)+W^2/2 \)

\( J \quad \text{station along x1 coordinate lines} \)

\( K \quad \text{station along x2 coordinate lines} \)

\( L \quad \text{distance along streamline} \)

\( q \quad \text{any fluid quantity} \)

\( r, \varphi, z \quad \text{angular momentum of fluid about axis of rotation} \)

\( W \quad \text{relative velocity} \)

\( W' \quad \text{contravariant component of W} \)

\( W_i \quad \text{covariant component of W} \)

\( W^2 \quad \text{physical component of W tangent to x^t} \)

\( t \quad \text{time} \)

\( x_1, x_2, x_3 \quad \text{general non-orthogonal coordinates} \)

\( \kappa \quad \text{ratio of specific heats} \)

\( \xi_{ij} \quad \text{angle included by the coordinate lines x^1 and x^2} \)

\( \varepsilon \quad \text{absolute vorticity, } \nabla \times V \)

\( \psi \quad \text{non-orthogonal coordinate system} \)

\( p \quad \text{gas density} \)

\( \omega \quad \text{stream function} \)

Subscripts

\( c \quad \text{casing} \)

\( h \quad \text{hub} \)

\( i \quad \text{inlet} \)

\( m \quad \text{mean (mid-channel)} \)

\( n \quad \text{component in the direction normal} \)
more solutions obtained by this method would mean a quicker, more streamlined series-expansion solution for axial compressor, but it is necessary to compute flow from Ref. (3-7), or by the approximate, but finite difference method (for example, early work Ref. (9,10)).

In cases where the assumption that $S_1$ surfaces are surfaces of revolution are not good enough, it is necessary to compute flow on general $S_1$ surfaces and a number of general $S_2$ surfaces, and solution for axial compressor by finite difference method and for centrifugal compressor by finite element method have been obtained according to this procedure. More solutions obtained by this method would define the area within quasi-3D flow is accurate enough for engineering application. Full 3-D solution of turbomachine flow may also be obtained without assuming that $S_1$ surfaces are surfaces of revolution namely determination by the use of Taylor series expansions, and the circumferential derivatives in the series are calculated from known values on the $S_2$ surface.

This procedure is different from the two-dimensional mean-streamline series-expansion method in two important respects. First, the circumferential derivatives are to be calculated from flow variation along one coordinate line other than in the circumferential coordinate line but also from the flow variation along the fluid coordinate as well. Second the successive $S_2$ surfaces in the circumferential directions cannot be determined simply by a mass balance on the surface of revolution as in the two-dimensional case. This type of 3-D solution was first obtained in Ref. (19, 20). A three-dimensional flow field can be obtained very quickly.

Being an approximate method, the method may be limited to a range of value of flow configuration and design parameters. It should be very useful in analysing the nature of 3-D flow corresponding to certain flow parameters and geometrical configurations, and especially, in a preliminary 3-D design blades of turbomachines.

**DETERMINATION OF THE SHAPE OF $S_m$ AND THE FLOW ALONG $S_m$**

The $S_m$ calculation is the same as that in a quasi 3-D design procedure. The hub and casing wall contours and the projection of the $S_m$ surface are specified by the designer. The variation of the angular momentum of the fluid about the machine axis (Ref.) along the $S_m$ surface and the circumferential thickness of the $S_m$ surface are also specified by the designer. The latter is to be obtained empirically from a desirable thickness distribution of the blade to be designed. Calculation of $S_m$ flow gives the shape of the $S_m$ surface, the streamline distribution on the $S_m$ surface, and the variation of all flow properties on the $S_m$.

**THE PARTIAL DERIVES OF FLOW QUANTITIES IN NONORTHOGONAL CURVILINEAR COORDINATES**

The flow properties at a point in the flow passage can be obtained from the known values on the $S_m$ surface by Taylor series expansion as follows:

$$q'(p) - q'(p_m) + (q - q_m) q'(p_m) + \frac{(q - q_m)^2}{2} q''(p_m)$$

In Ref. (5), the vorticities and continuity equations were expressed as follows:

$$\gamma' = \frac{1}{\sqrt{a}} \left[ \frac{\partial (q'p) - q_m}{\partial z} - \frac{\partial q_m}{\partial p} \right]$$

$$\gamma'' = \frac{1}{\sqrt{a}} \left[ \frac{\partial (q''p) - q_m}{\partial z} - \frac{\partial q_m}{\partial p} \right]$$

$$\gamma''' = \frac{1}{\sqrt{a}} \left[ \frac{\partial (q'''p) - q_m}{\partial z} - \frac{\partial q_m}{\partial p} \right]$$

\[1\]
When the flow is isentropic and irrotational, the first and second derivative of \( W, W_1, W_2 \) with respect to \( \xi \) can be obtained in a manner similar to that used in Ref.(1) as follows (For details, see Appendix I):

\[
\begin{bmatrix}
\sqrt{a_1 \cos \theta_1} \\
\sqrt{a_1} \\
\frac{1}{\sqrt{a_1}} \frac{\partial W}{\partial \xi}
\end{bmatrix} \frac{\partial}{\partial \xi} \begin{bmatrix}
\theta (V_1, x) \\
\theta (V_2, x) \\
\frac{\partial W}{\partial \xi}
\end{bmatrix} = C
\]

(4)

where

\[
C = \left[ \frac{\partial}{\partial \xi} \left( r \rho W \sqrt{u_0 \sin \theta_0} \right) + \frac{\partial}{\partial \xi} \left( r \rho W^{2} \sqrt{u_0 \sin \theta_0} \right) \right] - \frac{1}{r} \frac{\partial}{\partial \xi} \sqrt{a_1},
\]

(5)

On the \( S_{2m} \) surface, \( W_{n} (\phi = \phi_{m}) = 0 \).

(8)

Thus, all orders of partial derivatives of \( W_{n} \) on \( S_{2m} \) with respect to \( \phi \) at the hub and casing should be equal to zero, satisfying the solid wall condition. As a first order approximation, the first partial derivative should be equal to zero. In non-orthogonal curvilinear coordinate this relation is

\[
\nabla \cdot (\rho W) = 0
\]

(3)

By adjusting the variation of \( W_{n} \) on \( S_{2m} \) in the vicinity of hub and casing wall, the values of \( \frac{\partial W_{n}}{\partial \xi} \) can easily be modified to satisfy Eqs.(6). This method obtained through our numerical calculations is similar to the conclusion reached in Ref.(22). Table shows variations of \( \omega \overline{W_{n}} \) and \( V_{n} \) at the hub and casing boundaries on \( S_{2m} \) in our numerical calculation.

\[
\begin{array}{cccccccc}
\phi & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
W_{n} & 11 & 13.1 & 14.9 & 2.5 & -6.0 & -28.9 & -42.0 & -12.1 & 251.0 \\
V_{n} & 77.5 & 81.1 & 77.8 & -6.1 & -17.3 & 13.3 & -9.6 & 181.9 & 288.8 \\
\end{array}
\]

THE ANNULAR CONSTRAINT

It is shown in Ref. (21), when the integrability condition is considered in blade design, the flow variation a designer is free to specify is stream filament thickness, in addition to the variation of the thickness only one aerothermodynamic relation on the \( S_{2m} \) may be specified, which is usually the variation of \( W_{n} \).

A few computations by Eq.(1) immediately show that, the aerothermodynamic flow quantities must satisfy a constraint condition in order to guarantee that the \( S_{1} \) stream surfaces obtained on the casing and hub wall are surfaces of revolution. The velocity component normal to hub or shroud \( (W_{n}) \) should be equal to zero. As the same as other fluid properties, \( W_{n} \) should be obtained from the known values on the \( S_{2m} \) surface by Taylor series expansion:

\[
W_{n}(\phi) = W_{n}(\phi_{m}) + (\phi - \phi_{m}) \frac{\partial W_{n}}{\partial \phi} + \frac{(\phi - \phi_{m})^{2}}{2} \left( \frac{\partial^{2} W_{n}}{\partial \phi^{2}} \right) \]

(7)

On the \( S_{2m} \) surface, \( W_{n}(\phi = \phi_{m}) = 0 \).

Thus, all orders of partial derivatives of \( W_{n} \) on \( S_{2m} \) with respect to \( \phi \) at the hub and casing should be equal to zero, satisfying the solid wall condition. As a first order approximation, the first partial derivative should be equal to zero. In non-orthogonal curvilinear coordinate this relation is

\[
\nabla \times \nabla = 0
\]

(10)

Determination of the blade shape is the key part of this 3D mean-stream surface method. The blade shape cannot be determined only by the mass flow relation on a surface of revolution as in the 2D mean-streamline method. The following function proposed in Ref.(23) is used in Ref.(24) to describe the flow on the two intersecting families of \( S_{1} \) and \( S_{2} \) surfaces in a 3-D turbomachine flow passage. (see Fig.1)
where \( D_1, E_1, \ldots, D_n, E_n \) are values of stream functions on these stream surfaces. In the three-dimensional space, the distribution of these stream surfaces is restricted by the boundaries of the flow passage as follows:

1. Hub and casing walls are two stream surfaces of the \( S_1 \) family.
2. Pressure and suction surfaces of the blade are two stream surfaces of the \( S_o \) family.

From the known \( S_{2m} \) surface a family of \( S_{2n} \) surfaces will be formed (see next section). In this family, two surfaces are the pressure surface and suction surface of two neighboring blades, respectively, \( S_{2m} \) divides the total mass flow through the channel into suitable proportions.

**PORING \( S_2 \) FAMILY BY PROGRESSING CIRCUMFERENTIALLY FROM \( S_{2m} \)**

Similar to the 2-D mean-streamline method, the \( \psi \) coordinate of \( S_2 \) surface can be obtained by the use of a series expansion from \( S_{2m} \) as follows:

\[
\Delta \psi = \psi - \psi (\psi_0 - \psi_m) + \frac{1}{2} (\psi_0 - \psi_m)^2 + \cdots
\]

where \( \varphi \) is the circumferential coordinate of any \( S_2 \) surface and \( \psi_m \) is the value of stream function of that \( S_2 \) surface.

The integrating factor \( \beta \) of Ref. (1) is now expressed as follows (see Fig. 2)

\[
\frac{\Delta \psi}{\Delta \varphi} = \frac{\beta}{\beta_1} = \beta_0 + \beta_1 \frac{\partial \ln \beta}{\partial \varphi} + \frac{\partial \ln \beta}{\partial \varphi} = C
\]

\[
\frac{\beta}{\beta_1} = \frac{w_b}{w_1} \frac{\partial \ln \beta}{\partial \varphi} + \frac{\partial \ln \beta}{\partial \varphi} = C
\]

\[
\Delta \psi = (\psi - \psi_0)^{2} = \frac{1}{2} (\psi_0 - \psi_m)^2 + \frac{1}{2} (\psi_0 - \psi_m)^2 + \cdots
\]

From Eqs. (8) and (10),

\[
\frac{n}{n_1} = \frac{\Delta \psi}{\Delta \varphi} = \Delta \psi/\Delta \varphi
\]

When \( \Delta \psi \) approaches zero as a limit, \( B/B_1 \) becomes the first derivative of \( \varphi \) with respect to \( \psi \). Using first order expansion, the \( \varphi \) of \( S_2 \) surface can be determined step by step as follows:

\[
\Delta \psi = (\psi_0 - \psi_m) + \frac{1}{2} (\psi_0 - \psi_m)^2 + \frac{1}{2} (\psi_0 - \psi_m)^2 + \cdots
\]

When the \( \varphi \)-coordinate of \( (\psi + 1) \)th \( S_2 \) surface is obtained, the fluid properties \( W^f, W^s, \), and \( \beta \) on the \( S_{2m} \) surface can be determined by power series from \( S_{2m} \):

\[
W^f(\varphi) \cdot W^s(\varphi_m) + (\varphi - \varphi_m) \frac{W^s(\varphi)}{\partial \varphi} \varphi_m + \frac{1}{2} (\varphi - \varphi_m)^2 \frac{W^s(\varphi)}{\partial \varphi} \varphi_m + \cdots
\]

Subsequently, the Eq. (14) gives \( (B/B_1) \) and the position of the next \( S_2 \) surface can be determined.

**Fig. 2 Family of \( S_2 \) surface and intersection with inlet plane \( Z_1 \)**

The inlet plane \( Z_1 \) is usually sufficiently far upstream of the blade row, where the flow can be considered to be uniform. The intersection of the family of \( S_2 \) surface with this \( Z_1 \)-plane is a family of curves.

The variation of the stream function of two adjacent \( S_o \) surfaces is the mass flow passing through the small area between the two adjacent curves (shaded area in Fig. 2). Then the variation of \( \psi_0 \) can be determined in the inlet plane \( Z_1 \) as follows

\[
\psi_0 = \psi_0 (r, \varphi, z)
\]

\[
\psi_0 = \psi (r, \varphi, z)
\]

\[
\psi_0 = \psi (r, \varphi, z)
\]
The surfaces of revolution obtained by rotating around z-axis with the \( x' \) and \( x'' \) curves, and the \( S_2m \) stream surfaces are the new coordinate surfaces. The following relations may be obtained directly:

\[
\frac{\partial \eta}{\partial x'} = \frac{\partial \eta}{\partial x''} = \frac{\partial \eta}{\partial x} \quad (20)
\]

\[
\frac{\partial}{\partial \psi} \left( \frac{\partial \eta}{\partial x'} \right) = \frac{\partial}{\partial \psi} \left( \frac{\partial \eta}{\partial x''} \right) = \frac{\partial}{\partial \psi} \left( \frac{\partial \eta}{\partial x} \right) \quad (21)
\]

\[
W^1 = \tilde{W}, \quad W^2 = \tilde{W}, \quad W^3 = 0 \quad (22)
\]

where a bold partial derivative sign is used to denote the partial derivative of a quantity following the motion along \( S_2m \) with respect to \( x^1 \) or \( x^2 \) on the meridional plane. The \( W^1, W^2, W^3 \) are used to represent the contravariant components of \( W \) in \( (x^1, x^2, \psi') \) or \( (x^1, \eta, \varphi) \) system, respectively. The derivatives of \( W^1, W^2, \psi, \psi_0, \) and \( \varphi \) with respect to \( \psi_2 \) can be obtained. The details are shown in Appendix II. Differentiating the relation (14) with respect to \( \psi_2 \) and combining with relations (21) determines the first derivative of the angular thickness \( \beta \) with respect to \( 1/(\psi_2) \):

\[
\frac{1}{\beta} \frac{\partial (\delta \eta)}{\partial \psi_2} = \frac{1}{\beta^2 \sin \theta_0} \left[ \frac{\partial}{\partial \psi} \left( \frac{\partial (W^0)}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left( \frac{\partial (\psi_0)}{\partial \psi} \right) \right] \quad (23)
\]

The \( \varphi \) coordinate of any \( S_2 \) surface, especially those of the suction or pressure surface of the two neighboring blades, can be obtained directly from \( S_2m \) by a power series expansion

\[
\varphi = \varphi + \delta \psi + \frac{1}{2} \delta \psi + \frac{1}{6} \delta \psi + \ldots
\]

The fluid properties \( W^1, W^2, \) and \( \varphi \) on the suction or pressure surface can then be obtained by Taylor series:

\[
W^1(\psi) = W^1(\psi) + \frac{\partial W^1}{\partial \psi} \delta \psi + \frac{1}{2} \frac{\partial^2 W^1}{\partial \psi^2} \delta \psi + \ldots
\]

The distribution of \( S_1 \) stream function, \( \varphi \), in the flow passage can also be obtained by Taylor series, determines the shape of \( S_1 \) sur-

**ILLUSTRATIVE EXAMPLE**

FORTRAN IV programs based on the two methods presented herein have been coded to study the 3-D flow in axial turbomachines. The blade shape, the distribution of \( S_1 \) and \( S_2 \) surfaces, and the fluid properties in the flow passage can be obtained in less than a minute on a UNIVAC-1100 computer.

Example 1. The stator of a single-stage CAS research compressor reported in Ref. (16). The mass flow is 61 kg/sec, the rotor tip \( W \) is 1.4, the stage total-pressure ratio is 1.5. The hub-tip ratio at stator inlet is 0.49, the number of stator blades is 37. The projection of \( S_2m \) and its streamline distribution on the meridional plane is shown in Fig. 5.
The specified radial variation of \( V_r \) in front of the stator is shown in Fig. 6. Fig. 7 shows the blade thickness distribution specified. The blade shape and distribution of velocity on the \( k=7 \) coordinate surface obtained by the two methods are shown in Fig. 8. The difference between the two is small. A comparison of Mach Number distribution with that obtained in the full three-dimensional solution is shown in Fig. 9. Except near the leading edge, the result obtained from the present method is close to that of the 3-D solution. The relative twist of \( S_1 \) surfaces at the suction surface \((J=4)\) is shown in Fig. 10.

![PRESENT METHOD vs COMPLETE 3-D SOLUTION](image)

**Fig. 9 Distribution Mach Number on \( k=9 \) \( S_1 \) surface**

![PRESENT METHOD vs COMPLETE 3-D SOLUTION](image)

**Fig. 10 The relative twist quantity on the \( J=3 \) suction surface**

\((R)\) is the local value to the point \( J=4, k=1, \ldots, 11 \). \( R_m \) is the value of \( R \) on the \( S_{2m} \) surface of the same \( S_1 \) surface. It is seen that the largest \( \Delta R/R \) occurs a short distance from the hub wall and the maximum difference between the present solution and the full 3-D solution also occurs there. It may be noted that the maximum relative twist is rather small, being less than 1%.

**Example 2.** A turbine rotor. The mass flow is 2.2 kg/sec, the expansion ratio is 1.4, the rotor speed is 16000 rpm, the number of blades is 60, and the hub-tip ratio is 0.66. The meridional projection of \( S_{2m} \) and its distribution of streamline are shown in Fig. 11. The blade shape and the circumferential variation of velocity on \( k=3 \) coordinate surface obtained in the direct expansion method are shown in Fig. 12. The intersections of the \( J=3,4,5 \) coordinate surfaces with the \( S_1 \) surface and \( S_2 \) surface are shown in Fig. 13.

![Fig. 11 Meridional projection of \( S_{2m} \)](image)

**Fig. 11 Meridional projection of \( S_{2m} \)**

![Fig. 12 The distribution of \( W \) on \( k=3 \) coordinate surface](image)

**Fig. 12 The distribution of \( W \) on \( k=3 \) coordinate surface**

![Fig. 13 Intersection of \( S_1, S_2 \) surfaces with \( J=3,4,5 \) coordinate surfaces](image)

**Fig. 13 Intersection of \( S_1, S_2 \) surfaces with \( J=3,4,5 \) coordinate surfaces**

The direction of twist of the \( S_1 \) surface is basically consistent with the conclusion reached in Ref. (22); i.e., under most conditions, the \( S_1 \) surfaces twist upward at the suction surface in the turbine stator, and inward in the turbine rotor. As this turbine is designed for radially non-uniform work output so that the twist involved is relatively large, but is still under 2%.

**CONCLUDING REMARKS**

1. In prescribing a desired flow variation on the mid-channel hub to tip \( S_{2m} \) stream surface in 3-D blade design, it should be noted that prescribed variation should satisfy a constraint condition in order that the resulting blade to blade \( S_1 \) flow surfaces along the hub and casing walls are surfaces of revolution.
2. A new coordinate transformation, in which the \( S_2 \) stream surface is taken as a coordinate surface, leads to a new method for calculating partial derivatives of flow variables with respect to the stream function of \( S_2 \) surface. High order derivatives can be obtained and consequently high order series expansion can be used to compute the 3-D flow field.
3. This 3-D mean-streamsurface series-expansion method is different from the 2D mean-streamline series-expansion method in that the known flow variations along two general
curvilinear coordinates on the mean-streamsurface are now used to compute the circumferential derivatives of the flow variables. Based on the 3-D flow field obtained by series expansion, the blade shape corresponding to a mass flow rate passing through the 3-D blade channel can be determined. The blade to blade surfaces obtained in the solution between (and not including) the hub wall and casing wall are general S1 stream surfaces, and twists of these S1 surfaces are readily available in the calculation.

4. Two illustrative examples indicate that the method for solving the inverse problem of 3-D flow by the use of the present method is a feasible quick approximate engineering method. The nature and magnitude of the twist of S1 surface obtained in a high-subsonic axial-flow compressor and a high-subsonic axial-flow turbine are determined in the computation.

5. This simple, quick, approximate 3-D method should be useful to evaluate the nature and order of magnitude of the departure of blade-to-blade S1 surface from surface of revolution and in different geometrical configurations and design specifications, the accuracy of quasi-3D solution as compared to full 3D solution, and for 3D blade design, especially where a certain 3D distribution of blade thickness is desired. It may also be used as a 3D analysis method by successively correcting the mean-streamsurface flow. This correction is similar to the successive correction of the mean-streamline shape in the 2D analysis method.

REFERENCES


Consider now a unit vector normal to the S2m surface

\[ \mathbf{n} = n_i \mathbf{e}_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \]  

(1.1)

The orthogonal relation is expressed by

\[ \mathbf{n} \cdot \mathbf{e}_i \cdot \mathbf{e}_i = n_i n_j + n_j n_i = 0 \]  

(1.2)

The derivative of a fluid quantity on S2m with respect to \( x^1 \) and \( x^2 \) is

\[ \mathbf{\frac{\partial}{\partial x^1}} \mathbf{\frac{\partial}{\partial x^2}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(1.3)

\[ \mathbf{\frac{\partial}{\partial x^1}} \mathbf{\frac{\partial}{\partial x^2}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(1.4)

\[ \mathbf{n} \cdot \mathbf{W} = 0 \]  

(1.5)

When the flow is irrotational the substitution of Eqs.(1.3, 1.4) into Eqs.(2a, b) gives

\[ \sqrt{a_n} \cos \theta_0 \mathbf{\frac{\partial}{\partial \theta_0}} + \mathbf{\frac{\partial}{\partial \theta_0}} \mathbf{\frac{\partial}{\partial \theta_0}} = \mathbf{\frac{\partial}{\partial \theta_0}} \mathbf{\frac{\partial}{\partial \theta_0}} \]  

(1.6)

\[ \sqrt{a_n} \mathbf{\frac{\partial}{\partial \theta_0}} + \mathbf{\frac{\partial}{\partial \theta_0}} \mathbf{\frac{\partial}{\partial \theta_0}} = \mathbf{\frac{\partial}{\partial \theta_0}} \mathbf{\frac{\partial}{\partial \theta_0}} \]  

(1.7)

When Eqs.(1.3, 1.4, 1.5) is used, Eqs(3) can be written as

\[ \mathbf{\frac{\partial}{\partial x^1}} \mathbf{\frac{\partial}{\partial x^2}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(1.8)

Differentiating Eqs.(1.6-1.8) with respect to \( \theta \) and combining with Eqs.(1.2-1.4) give the second derivatives of \( W, W_2, W_4 \) as

\[ \mathbf{\frac{\partial}{\partial x^1}} \mathbf{\frac{\partial}{\partial x^2}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(1.9)

where

\[ \mathbf{D} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} + \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(2.1)

\[ \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(2.2)

\[ \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(2.3)

The second derivatives of \( W, W_2, W_4, P \) can be obtained in a similar way.

From Eqs.(2.10)

\[ \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(2.4)

\[ \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(2.5)

Differentiating the Eqs.(2.5) with respect to \( \theta \) the first, second or more high order derivatives of \( \psi \) can be obtained:

\[ \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(2.6)

\[ \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(2.7)

étc.

Differentiating the Eqs.(2.3) with respect to \( \theta \) and combining with Eqs.(2.1) the second derivative of \( B \) is obtained.

\[ \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \]  

(2.8)

High order derivatives can be obtained by similar manner.

APPENDIX II

Substituting \( \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} = \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \mathbf{\frac{\partial}{\partial x^3}} \) into vorticity and continuity equation gives the derivatives of \( W, W_2, W_4 \).