Modelling of Unsteady Transitional Boundary Layers

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ABSTRACT

In turbomachinery, a considerable proportion of the blade surface area can be covered by transitional boundary layers. This means that accurate prediction of the profile loss and boundary layer behaviour in general depends on the accurate modelling of the transitional boundary layers, especially at low Reynolds numbers. This paper presents a model for determining the intermittency resulting from the unsteady transition caused by the passage of wakes over a blade surface. The model is founded on work by Emmons (1951) who showed that the intermittency could be calculated from a knowledge of the behaviour of randomly formed turbulent spots.

The model is used to calculate the development of the boundary layer on the rotor of a low Reynolds number single-stage turbine. The predictions are compared with experimental results obtained using surface-mounted hot-film anemometers and hot-wire traverses of the rotor mid-span boundary layer at two different rotor-stator gaps. The validity and limitations of the model are discussed.

INTRODUCTION

The successful design of turbomachine blading requires accurate and reliable predictions of boundary layer development. When the boundary layers are either laminar or turbulent, the prediction process is relatively straightforward. In reality, however, a considerable area of a blade surface can be covered by a boundary layer which is only intermittently turbulent. This is especially true at the low Reynolds numbers which are typical of the low-pressure stages in gas turbines.

Start of transition and transition length correlations (e.g. Abu Ghannan and Shaw, 1980) can be used to determine the extent and location of the transitional region when the flow is steady and two dimensional. However, the flow within a gas turbine stage is known to be unsteady. In particular, the wakes from upstream blade rows can promote transition ahead of that occurring for clean inlet flow. Hodson (1983) showed that this could increase the profile loss generated by a turbine rotor suction surface boundary layer by 50%. The rate of heat transfer may be similarly effected (e.g. Doorly, 1983). Traditionally, the effect of wake interactions is seldom taken into account in design calculations other than by assuming that the wake turbulence "averages out" to some mean level so that the problem reduces to one of computing a steady flow.

Only recently has attention turned to the problem of calculating the unsteady transitional behaviour and its effects (e.g. LaGriff et al., 1988; Sharma et al., 1988; Hodson, 1989; Mayle and Dullenkopf, 1989; Mayle and Dullenkopf, 1990). The present paper presents a model for the development of the transitional region resulting from the passage of wakes shed by an upstream blade-row.

The Mechanics of Boundary Layer Transition

The basic stages in the process of boundary layer transition are now reasonably well understood even if some of the details are not. A full discussion of the steady transition process is beyond the scope of this paper. Reviews can be found in Morkovin (1958), Arnal (1984) and Narasimha (1985).

Transition starts with the amplification of disturbances within the boundary layer transferring energy from the mean flow to the disturbances. These disturbances can come from a number of possible sources. When there is little or no free-stream turbulence, the first stages of amplification are frequency selective and Tollmien-Schlichting waves appear. For a free-stream flow with a higher turbulence intensity, which is much more typical of the turbomachinery environment, large amplitude disturbances diffuse directly into the boundary layer thus by-passing the initial Tollmien-Schlichting amplification process. Transition then occurs directly from a background of disturbances in the laminar boundary layer.

Once the amplitude of the disturbances reaches some critical level (e.g., u'/U=0.2 in the work of Elder, 1960), the flow breaks down and localized spots of turbulent flow appear. These spots were first observed by Emmons (1951). Schubauer and Klebanoff (1955) who used sparks to induce turbulent spots in a laminar boundary layer at specific intervals, found that the spots are roughly triangular in plan view although their shape varies somewhat across the height of the boundary layer as shown in fig. 1a. Since that time, a number of other investigations have looked at the behaviour of turbulent spots and, although conditions can affect the details, their basic nature remains as that shown in fig. 1a.

Emmons suggested that the rate of formation of spots was uniform in space after the start of transition but later work by Narasimha (1957) showed that the spots formed only in a region close to the start of transition which was narrow in length compared to the length of the subsequent transitional region. This is because upstream of the start of transition, spots are unable to form while downstream of the start, the formation of spots is inhibited by becalmed regions following spots which were formed earlier. The formation of the turbulent spots occurs randomly across the span but at a single fairly well defined height within the boundary layer. This most often happens nearest the surface. For the purposes of the calculation of transitional boundary layers Narasimha thus proposed that, as a first approximation, the start of transition could be considered to be a span-wise line as shown in fig. 2, where the function g represents the formation rate of spots per unit time per unit volume.

The turbulent spots grow as they convect in the stream-wise direction so that the leading edge and trailing edge velocities of the spot are typically 88% and 50% Ue respectively. The span-wise spreading
time, the boundary layer may be laminar or turbulent. The probability that it will be turbulent increases towards the rear of the transition zone. The intermittency $\gamma$ is this probability. It is equal to zero for laminar flow and is unity for turbulent flow. Note that this definition presumes that the flow is either laminar or turbulent. In steady flow problems, the intermittency is usually taken to be the fraction of the time that the boundary layer will be turbulent at that particular point in space. However, the more rigorous definition is required for work of the type reported in this paper where the start of transition oscillates back and forth along the blade surface in sympathy with the passing of the upstream wakes over the blade surface. Once the intermittency has been specified, it is a relatively simple matter to calculate the flow through the transition zone.

**Transition in Turbomachinery**

Transition in turbomachinery is often induced by the passing of wakes from an upstream blade-row. Work by Pfeil et al (1982) and subsequent workers (e.g. Dooley, 1983; LaGraff et al, 1988) suggested that this manifested itself as the formation of span-wise bands of turbulent flow beneath the wakes which then spread as if they were large one-dimensional (stream-wise) turbulent spots. Because of the difference between the leading and trailing edge velocities, adjacent bands would merge with each other and form a fully turbulent boundary layer. This hypothesis was used as a basis for the modelling work of LaGraff et al (1987) and Mayle and Dullenkopf (1989 and 1990) with some success. The latter work of Mayle and Dullenkopf showed, like Hodson (1989) that this hypothesis works well at high Reynolds numbers and the experimental data could be correlated using a new form of the reduced frequency parameter (see Hodson, 1989). Other recent work, however, (e.g. Addison and Hodson, 1989; Hodson 1989) has shown that the above hypothesis is unable to explain the distribution of turbulent flow on the surface of low Reynolds number blading.

Addison and Hodson (1989) proposed the physical model which is depicted in fig. 3. When a wake passes over a laminar boundary layer, disturbances rapidly diffuse into the boundary layer leading to the formation of a band of disturbed laminar flow beneath the wake which extends across the span (fig. 3(a)-(c)). The shaded contours indicate the intensity of these disturbances. They are associated with low Reynolds stresses and have little effect on the boundary layer profile. As the boundary layer thickness increases downstream, the boundary layer becomes more susceptible to transition. Turbulent spots then begin to form randomly (fig. 3(d)). These occur earliest not at the centre-line of the disturbed laminar patch where the intensity of the disturbances is highest but further along the blade where the balance between the production and dissipation of
turbulent kinetic energy swings in favour of production. A little later (fig. 3(e)), the wake has moved further along the surface and the spots now form nearer to centre-line of the wake. In this case, the wake is sufficiently broad that the start of transition line has moved toward the leading edge. The remaining plots (fig. 3(f)-(h)) show how the start of transition line then retreats but at a slower rate than the wake. Thus, the start-of-transition line oscillates back and forth in a rather complex fashion, the precise nature of the oscillation being governed by the combined effects of the intensity and duration of the free-stream disturbances, the local boundary layer thickness and the local pressure gradient.

Although, at first sight, it may appear that the above model conflicts with that presented by, for example, Mayle and Dullenkopf (1989, 1990) or Hodson (1989), it was assumed that the wake induced turbulent patch spread across the entire span, this is not the case. In the case of higher Reynolds numbers, the intrinsic disturbances are much less inside the boundary layers so that the spots appear always to form much nearer to the centre-line of the wake. If the spots form only at the centre-line of the wake, then the start of transition line moves at the wake speed, creating turbulent spots which lie just ahead of those created earlier whose leading edge propagation speed is approximately 0.88 \( U_e \). As a result, the spots rapidly coalesce and a band or strip of turbulent flow results whose leading and trailing edge propagation velocities are respectively 1.0 and 0.5 \( U_e \) as proposed by Hodson (1989).

The model described above does not conflict with what is known about the origins of turbulent spots in steady flow. In practice, this means that wake induced transition can simply be treated as transition resulting from a moving start of transition line where spots are formed in the same way as for steady transition, the motion of the line being defined by the behaviour of the disturbances originating in the wake. The problem of the prediction of the transition process can now be divided into two parts:

1) the prediction of the location of the (unsteady) start of transition,

2) the prediction of the stochastic formation and subsequent growth of the turbulent spots.

The model to be presented in this paper is specifically concerned with the development of the transition region, the start of transition having been determined from experimental data.

A MODEL FOR THE DETERMINATION OF THE INTERMITTENCY IN UNSTEADY TRANSITIONAL BOUNDARY LAYERS

The model outlined in this paper depends on the concept of the volume of dependency \( R \) of a point \( P(x,z,t) \) which was originally proposed by Emmons (1951). This is the volume in space and time which contains all points which could have been the origin of a turbulent spot which subsequently passed over point \( P \). Emmons showed that the probability that the flow was turbulent at the point \( P \) was given by

\[
\gamma(P) = 1 - \exp \left[ - \iiint_R g(P_0) \, dV_0 \right]
\]

where \( g(P_0) \) is the number of spots per unit time formed in an elemental volume \( dV_0 \) at point \( P_0(x_0,z_0,t_0) \) within \( R \). Eqn. 1 means that the intermittency can be determined if the spot formation parameter \( g(P_0) \), which governs the rate and location of formation of turbulent spots, and the geometry of the dependence volume \( R \), which is defined by the spreading behaviour of the individual turbulent spots, can be specified.

It will be assumed that there is no variation in behaviour normal to the surface (y-direction) even though, as was noted above, the shape of the turbulent spot varies with height through the boundary layer. Dhawan and Narasimha (1958) and many others have shown that successful computations can be made when this assumption is employed. It is, of course, essential when using integral boundary layer solvers. Similarly, the model ignores the bicalmed regions just behind the spots in which the boundary layer is reverting to the laminar profile. However, the errors involved are believed to be relatively small and, consequently, the added complexity which incorporation would involve is unnecessary.

As mentioned above, Narasimha (1957) showed that turbulent spot formation occurred in a narrow span-wise strip around the start of transition line (see fig. 2). He also showed that when spots are only formed along the start of transition line, Eqn. (1) becomes

\[
\gamma(P) = 1 - \exp \left[ - \iiint_A n(s_{tr}) \, dz_0 \, dt_0 \right]
\]

where \( s_{tr} \) is the stream-wise location of the start of transition point (the point at which breakdown and the creation of turbulent spots occurs) and \( n(s_{tr}) \) is the formation rate of turbulent spots per unit length in the span-wise direction along the start of transition line. Ashworth (1987) and LaGraff et al (1988) used a Gaussian distribution for the spot formation parameter \( g(s) \). However, as was shown by Dhawan and Narasimha (1958), the differences between the resulting intermittency distributions are relatively small for most practical purposes.

![Fig. 4](https://example.com/f04.pdf)

**FIG. 4** THE DEPENDENCE VOLUME OF POINT P AND THE SIMPLIFICATIONS INTRODUCED IN THE PRESENT MODEL

The geometry of the volume of dependence is determined by the shape of the turbulent spots as they grow and spread downstream. Practical implementation of the model is considerably simplified by following the work of Chen and Thyson (1971), thereby assuming that the plan view of a typical turbulent spot is an isosceles triangle with a half-angle of about 15° and leading and trailing edge speeds of 0.88 and 0.5 \( U_e \) respectively (fig. 1b). These assumptions result in the volume of dependence shown schematically in fig. 4. Narasimha (1958) gives numerous examples of how the values of propagation velocities and spot wedge angles given above vary with Reynolds number, pressure gradient, Mach number and so on. In addition, Ashworth (1987) has experimented with a more accurate representation of the spot geometry. However, for the sake of simplicity, none of these details have been included in the present model.

To complete the model, an estimate for the rate of formation of turbulent spots \( n(s_{tr}) \) is required. Narasimha (1958) presented data on the spot formation rate in the form of the non-dimensional group

\[
N = \frac{\sigma D_s^3}{\theta} \frac{n(s_{tr})}{v}
\]

where \( \theta \) is the boundary layer momentum thickness at the start of transition and \( \sigma \) is known as the spot area dependence factor. Estimates
for this latter value vary considerably, Emmons (1951) used \( \sigma = 0.1 \) in his original work and Narasimha (1985) quotes values ranging from 0.25 to 0.29 depending on the height within the boundary layer. A value of \( \sigma = 0.25 \) will be assumed for the work reported here.

Taking the measured intermittency distributions of a number of workers, Narasimha derived the value of \( N \) for a range of free-stream turbulence intensities. His results are shown in fig. 5. This work suggested that \( N \) had an almost uniform value for turbulence induced transition of \( N_t = 0.7 \times 10^{-3} \) for levels of free-stream turbulence intensity above \( T_u > 0.5\% \) although no details were given about the pressure gradients for each point. More recent research by Gostelow and his co-workers (Gostelow and Blunden (1988), Gostelow (1989) and Walker and Gostelow, 1990 and Gostelow and Dey, 1990) has shown that there is a considerable increase in the value of \( N \) for adverse pressure gradients. An attempt to correlate the available data against the pressure gradient parameter

\[
\lambda_\theta = \frac{\theta^2 \frac{dU_c}{d\bar{s}}}{u}
\]  

measured at the start of transition is shown in fig. 6. For the purposes of the modelling reported here, the following values will be assumed

\[
N = \begin{cases} 
N_t & \lambda_\theta \geq 0 \\
N_t \exp(-71.08\lambda_\theta) & \lambda_\theta < 0 
\end{cases}
\text{ where } N_t = 0.7 \times 10^{-3} \tag{5}
\]

By this action, the dependency of the parameter \( N \) on the level of free-stream turbulence is ignored even though, as fig. 6 shows, there is some dependency. For the present investigation, the free-stream turbulence intensities are of the order of 2 percent in the region of transition so that the correlation should be adequate.

A METHOD FOR THE CALCULATION OF UNSTEADY TRANSITIONAL BOUNDARY LAYERS

Implementation of the model requires the calculation of the boundary layer prior to, during and beyond the transitional zone. An adaptation of the Cebeci-Smith boundary layer code (described in Cebeci and Carr, 1978) was employed in the present work. As has already been noted above, the variation in time of the location of the start of transition must be specified. A laminar calculation then provides values for the momentum thickness of the boundary layer \( \theta_f \) at the start of transition for each instant in time. Once this has been determined, the non-dimensional pressure gradient \( \lambda_\theta \) may be determined. The value of \( N \) is then given by eqn. (5) and the dimensional spot formation rate \( n(s_\theta) \) can then be calculated using eqn. (3). The unsteady intermittency \( \gamma(s,t) \) can then be calculated for each point in space and time by numerical integration of eqn. (2) over the domain shown in fig. 4.

The unsteady intermittency itself is only of limited interest. In order to derive values which would be of more direct interest to designers, a time-mean intermittency \( \overline{\gamma}(s) \) was calculated and fed back to the boundary layer code. The problem is thus reduced to the calculation of a steady flow boundary layer with a prescribed intermittency distribution. The differential boundary layer code employs the Cebeci-Smith eddy viscosity formulation. For a given height within the boundary layer, the eddy viscosity was determined according to the relationship

\[
\nu_{\text{eff}} = \nu + \overline{\gamma}(s^*) \nu_t
\]  

where \( s^* \) is the non-dimensional surface distance, the eddy viscosity \( \nu_t \) is calculated as usual and the intermittency lies in the range 0 \( \leq \overline{\gamma} \leq 1 \). From this, the time-mean boundary layer parameters can be calculated.

It is recognized that the use of a time averaged intermittency function together with a steady flow prediction code denies the very existence of unsteady transitional flow. The justification of the use of a steady flow code is based on the authors' previous observations that the development of transitional boundary layers appears to be dominated by the turbulent fluctuations in the free-stream rather than by the ordered periodic fluctuations associated with the velocity defect in the wake. Hodson (1989) has also shown that there is very little difference between the approach presented here and one in which the development of individual packets of fluid which undergo transition at different points in time and space is calculated, albeit with a steady flow code, before averaging in time. Mayle and Dullenkopf (1989) have also demonstrated that the use of a time-mean intermittency gives satisfactory results. Given that transition in steady flows is an unsteady process, these observations may not be surprising.

COMPARISONS OF PREDICTIONS WITH MEASUREMENTS

Experimental Details

The predictions presented in this paper are to be compared with data obtained in the large scale low speed single stage axial turbine at the Whittle Laboratory, Cambridge University. The turbine and the data employed in the present study have been previously described, for

The stage was a free-vortex design with zero inlet and exit swirl and 50 percent reaction at mid-span. The 36 stator blades and 51 rotor blades had aspect ratios of 1.5 and 2.0 respectively and a hub-tip ratio of 0.7. The present study is concerned specifically with the rotor mid-span blade section, which was designed for relative inlet and exit angles of 0° and -65° respectively, corresponding to a flow coefficient \( V_s/U_b \) of 0.49. The flow at rotor mid-span was essentially two-dimensional (Hodson, 1983) and free from the influences of tip leakage and secondary flows (Hodson and Addison, 1988). Experimental data have been obtained at three stator-rotor axial gaps, corresponding to 50%, 75% and 143% \( C_s \) (stator axial chord) but only data obtained at the larger gaps will be referred to here. The different clearances were obtained by the use of spacer rings. Experimental data are also available from a linear cascade which consists of 7 blades with the same profile as the rotor mid-span section.

Hot-film gauges have been used to investigate the boundary layer behaviour of the rotor suction surface at the different axial spacings. At axial spacings of 75% \( C_s \) and 143% \( C_s \), Hodson (1984) used a single, calibrated hot-film gauge which was positioned at several locations along the blade surface, the data being logged at the rate of 10 samples per wake passing cycle and averaged over 100 revolutions.

Hot-wire traverses of the rotor blade boundary layers were originally reported by Hodson (1983). Those presented here were carried out at axial gaps of 75 and 143% \( C_s \) using a single hot-wire aligned perpendicular to the mean flow and parallel to the surface. Although ensemble averaged results have been presented by Hodson (1983) and Addison and Hodson (1989), only time mean measurements will be presented here.

**Predictions**

The velocity distribution for the mid-span section is shown in fig. 7. In this figure and elsewhere, the surface distance coordinate is measured from the predicted location of the stagnation point of the rotor blade and is expressed as \( s^* \), the ratio of the distance from the stagnation point to the length of the surface. The data were obtained from static pressure measurements in the equivalent linear cascade operating at an inlet turbulence intensity of 9.5%. At this condition, there is no separation bubble on the suction surface of the cascade so that the measured velocity distribution may be regarded as equivalent to the situation in the rotor where the suction side boundary layer does not separate.

The solid line shows the velocity distribution employed for the purposes of the present calculations. Detail around the leading edge was provided by a Martens calculation (see Wilkinson, 1968) while measured values were used from 37% \( s^* \). This gave the most acceptable distribution of the velocity and its gradients. The figure shows that the profile is moderately affected with the maximum velocity occurring near 45% \( s^* \) on the suction surface.

Having specified the external flow velocity \( U_{\text{ext}} \), a boundary layer calculation was performed and the momentum thickness \( \theta \) and pressure gradient parameter \( \lambda_0 \) was determined. From these and using eqn. (5), the spot formation rate \( n(t_s-t) \) the start of transition was obtained. Fig. 8 shows the variation of the predicted pressure gradient and of the spot formation rate for the rotor suction surface. The latter quantity has been normalized with the total stream-wise surface distance \( s_{\text{total}} \) and the blade passing time \( T \). The approximate extent of the unsteady transition zone is also indicated in the figure. The lower plot of fig. 8 shows that in the regions of adverse pressure gradient (i.e. \( \lambda_0<0 \)) towards the rear of the suction surface surface, there are considerable increases in the formation rate of the turbulent spots when compared with the formation rate given by assuming a constant rate (i.e. \( N_t=0.7\times10^{-3} \) which is indicated by the broken line. It should be noted here that few static pressure tappings were located in the transition region. It will be shown below that the lack of detail in the velocity distribution derived from the measured data can lead to errors in the predictions of the flow in these regions of diffusion.

The surface distance-time \((s^*-t^*)\) diagrams of fig. 9 are taken from the work of Addison and Hodson (1989). The two plots correspond to axial gaps of 75 and 143% \( C_s \). The contours are of the ensemble (phase-lock-averaged) root-mean-square of the random unsteadiness of the wall shear stress (i.e. \( \tau_w \)). Thus, the contours represent the intensity of the random unsteadiness.
The root-mean-square therefore represents the sum of the unsteadiness associated with a random change of state caused by the stochastic formation of the spots and the turbulence within the spots themselves. The relatively large values of the peaks are consistent with the fact that much larger changes of heat transfer rate result from a change of state than from the fluctuations observed in turbulent boundary layers. Once transition is complete, the root-mean-square corresponds to the level which is typical of that measured in turbulent boundary layers. A similar though less pronounced feature can also be seen in the time-averaged results of Pfeil et al. (1982).

The behaviour of the start of transition point $s_T$ was assumed to be that shown by the solid line in each of the surface distance-time ($s^*$-*t*) diagrams of fig. 9. These lines indicate the oscillating location of the start of transition predicted by Addison and Hodson (1989). Using this information, the unsteady intermittency $\gamma(s,t)$ was calculated for each point in space and time by numerical integration of eqn. (2) as described above.

![Fig. 9](http://manufacturingscience.asmedigitalcollection.asme.org/GT/proceedings-pdf/GT1991/78989/V001T01A090/2400617/v001t01a090-91-gt-282.pdf)

**Fig. 9** Definition of the unsteady start of transition point for the two rotor-stator axial spacings (Predicted start of transition compared with surface mounted hot film gauge measurements - data originally presented by Addison and Hodson, 1989) (a) 75% Cxs Rotor-Stator Axial Gap, (b) 143% Cxs Rotor-Stator Axial Gap

Whether this is due to the turbulent fluctuations, other random events or a combination of both.

Boundary layer traverse data (Hodson, 1983) shows that in the case of the 75% Cxs axial gap, transition has begun by 50% $s^*$ and extends to about 80% $s^*$. In fig. 9, the relatively low levels of random unsteadiness indicate the insignificance of the relatively low frequency (i.e. large scale) fluctuations in the disturbed laminar boundary layer over the first part of the blade. At the rear of the surface, where the boundary layer is turbulent, the values are slightly greater than during the laminar phase. Some of the energy in the turbulent boundary layers occurs at frequencies beyond the cut-off frequency of the instrumentation and so the values are artificially low in this region. Most significantly, the upper plot of fig. 9 shows that through the transitional region, the contour heights reach their maximum values. This is also true at the larger axial gap, where transition begins significantly later but was found to be complete by about the same location.

If the transition to turbulent flow were to have taken place in a phase-locked, periodic manner whether via the formation of a single spot or multiple spots then the levels indicated in fig. 9 would rise monotonically to the turbulent values during transition. The fact that they do not and the fact that the turbulent shear stress is much greater than the laminar value suggested to Addison and Hodson (1989) that the formation of individual spots which grow to form the transition wedges, is a stochastic process. This conclusion lead to the present study. The root-mean-square therefore represents the sum of the unsteadiness associated with a random change of state caused by the stochastic formation of the spots and the turbulence within the spots themselves. The relatively large values of the peaks are consistent with the fact that much larger changes of heat transfer rate result from a change of state than from the fluctuations observed in turbulent boundary layers. Once transition is complete, the root-mean-square corresponds to the level which is typical of that measured in turbulent boundary layers. A similar though less pronounced feature can also be seen in the time-averaged results of Pfeil et al. (1982).

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![Fig. 10](http://manufacturingscience.asmedigitalcollection.asme.org/GT/proceedings-pdf/GT1991/78989/V001T01A090/2400617/v001t01a090-91-gt-282.pdf)

**Fig. 10** Predicted time-mean intermittency on the turbine rotor for the two rotor-stator axial spacings

Fig. 10 shows the predicted time mean intermittency $\gamma$ as a function of surface distance $s^*$ for the turbine rotor suction surface at the different axial gaps. The solid lines refer to the model and start of transition criteria described above (see fig. 9). Two other predictions are also shown. The second prediction for the 143% Cxs which is shown as a broken line was obtained by assuming that the dimensionless spot formation rate $N$ had the constant value $N_0=0.7x10^{-3}$ given by eqn. (5). The hot-film data of fig. 9 indicate that the random unsteadiness has begun to rise prior to the start of transition lines shown in fig. 9. Since this might indicate that transition has already begun, a further prediction for the 75% Cxs test case was obtained. This is also shown as a broken line in fig. 10. This latter prediction was obtained by the simple expedient of subtracting 10% $s^*$ from the start of transition line shown in fig. 9. It will be used to study the effects of changing the location of the start of transition.

The intermittency distributions indicated by the solid lines in fig. 10 show that the time-averaged intermittency is predicted to be about 95% at the trailing edge when the axial gap is 75% Cxs. Transition begins later at the larger axial gap. This means that the start of transition is in a region of adverse pressure gradient and, consequently, the spot formation rate is much increased (see fig. 8). This, in turn, leads to a much shorter transition length so that the intermittency at the trailing edge is predicted to be about 90%, which is very similar to the situation at the smaller axial gap. The shape of the predicted intermittency distributions, as shown by the solid lines and the variation with axial gap is generally consistent with the experimental evidence provided by the hot-wire traverse data already referred to and the hot-film data of fig. 9.

Fig. 10 also serves to highlight the perils of neglecting the
increase in spot formation rate with pressure gradient. The predicted intermittency distribution for the larger axial gap which was obtained by specifying a constant value of $N$ indicates that the intermittency at the trailing edge is less than 50%. This is much lower than the 90% predicted above. Where transition takes place in an adverse pressure gradient such as would probably be the case on compressor blades or low pressure turbine blades, the correct specification of the spot formation rate could become a crucial factor in determining whether the suction side blade boundary layer will or will not separate.

The measured root-mean-square of a process which swings intermittently between two states with different mean levels will be greater than the root-mean-square measured for either one of the states. Fig. 11 shows the predicted wall shear stress distributions for the case of laminar flow and for the case when laminar flow ends at 30% $s^*$ where it is followed by fully turbulent flow. This location is upstream of the start of transition lines shown in fig. 9. The predictions in fig. 11 indicate that after transition, the difference in the friction coefficients is approximately constant. In practice, this remains true for very different locations of the start of transition.

![Graph of predicted time-mean skin friction coefficients for steady state laminar and laminar-turbulent boundary layers](image)

**Fig. 11** Predicted time-mean skin friction coefficients for steady state laminar and laminar-turbulent boundary layers

In the appendix, a functional relationship is derived between the expected root-mean-square $\sigma$ of the intermittent hot-film signal and the value of the intermittency $\gamma$. This relationship may be written as

$$\sigma^2 = (1-\gamma) \sigma_{\text{lam}}^2 + \gamma \sigma_{\text{turb}}^2 + \chi(1-\gamma)(\sigma_{\text{lam}} - \sigma_{\text{turb}})^2 \quad (A7)$$

while the mean level of wall shear stress $\tau$ is given by

$$\tau = (1-\gamma) \sigma_{\text{lam}} + \gamma \sigma_{\text{turb}} \quad (A8)$$

where $\sigma_{\text{lam}}$ and $\sigma_{\text{turb}}$ are the laminar and turbulent shear stresses respectively and $\sigma_{\text{lam}}$ and $\sigma_{\text{turb}}$ are the respective root-mean-squares. Thus, given the model of transition described and the value of intermittency, it is possible to predict the root-mean-square of the fluctuations within the transitional zone. In practice, $\sigma_{\text{lam}}$ and $\sigma_{\text{turb}}$ are very much less than $(\sigma_{\text{lam}} - \sigma_{\text{turb}})$ which is approximately constant (see fig. 11) so that the root-mean-square, in effect, becomes a parabolic function of the intermittency $\gamma$ with a maximum value when $\gamma = 0.5$.

In order to obtain the results which follow, the ratios $\sigma_{\text{lam}}/\sigma_{\text{turb}}$ and $\sigma_{\text{turb}}/\sigma_{\text{turb}}$ are assumed to be equal to the values observed upstream and downstream of the transition zone respectively. Using data similar to that shown in fig. 11, the root-mean-square intensity $\sigma/\tau$ was then calculated for each of the test cases. To obtain reasonable agreement with the measured data, it was also necessary to reduce by a factor of two the contribution to the rms of the last term in eqn. (A8). This is believed to be justified because the upper frequency response of the instrumentation would lead to too low a measured value and because upstream of transition, the measured mean values of the wall shear stress are approximately 30 percent in excess of the laminar values (Hodson, 1984) so that the measured root-mean-square intensity

$$\overline{Tu} = \frac{\sqrt{\langle \tau_{\text{turb}}^2 \rangle}}{\tau_{\text{lam}}}$$

which will have a maximum value proportional to $(\sigma_{\text{lam}} - \sigma_{\text{turb}})/\tau$ will have been reduced.

![Graph of predicted time-mean rms intensity compared to hot film gauge measurements](image)

**Fig. 12** Predicted time-mean rms intensity compared to hot film gauge measurements

(a) Rotor-Stator Axial Gap - 25% CxS
(b) Rotor-Stator Axial Gap - 143% CxS

Fig. 12(a) compares the predicted variations of the time-mean rms intensity $\sigma/\tau$ with the measured variation of $\overline{Tu}$ for the smaller axial gap. The two predictions correspond to the intermittency distributions shown in fig. 10. It is clear that in both cases, the position of the peak value, which is indicative of the region of maximum stochastic activity, can be seen to be reasonably well predicted. However, it is also clear that in both cases, the length of the predicted transition zone is a little longer than that indicated by the measured data. Fig. 12(b) compares the predicted and measured distributions at an axial gap of 143% $C_{xS}$. In this case, it is clear that transition, as specified in the predictions, begins too late. The measurements show that length of the transition zone is shorter than in the case of the smaller axial gap and the predictions accurately reflect this. The general agreement between the predictions and measurements presented in fig. 12 is taken as confirmation, albeit indirect, that the predicted intermittency distributions are realistic.

The experimentally determined surface distributions of the integral parameters of the suction surface boundary layer are also available for the 75% and 143% $C_{xS}$ test cases (see Hodson, 1983). Fig. 13 shows the measured variation of the momentum thickness $\theta$
and so have not been presented. Most significantly, the predicted momentum thicknesses at the trailing edge is 30 percent less than that measured. While the some of the differences may be due to experimental error, the fact that the results of fig. 12 have already shown, the predicted transition begins too late and the fact that the predicted transition length is too long cannot be ignored. In this test case, transition takes place in a diffusing flow. This is important because the exact value of the spot formation rate \( \beta \) is dependent upon an exact value for the pressure gradient parameter \( \alpha \) and the accuracy of the correlation shown in fig. 8 and given by eqn. (5).

Unfortunately, specifying an earlier start to transition places the start of transition in a region of relatively constant pressure so that the rate of spot formation is lowered. In turn, this means that the transition length is predicted to be far too long (the intermittency hardly reaches 50 percent at the trailing edge) and this leads to an even greater error in the predicted momentum thickness. The predictions corresponding to the constant rate of spot formation also serve to highlight the fact that the results produced by calculations of this type are extremely dependent upon the correct specification of the spot parameters. Not only does this require further research but also an accurate and detailed knowledge of the blade surface pressure distribution because these spot parameters are in part determined by the velocity gradients at the start of transition.

**Further Discussion**

The model presented above and its use to predict the growth of a boundary layer on a turbine rotor blade has served to highlight a number of problems. In particular, the significance of determining an accurate value for the increased spot formation rate in adverse pressure gradients has been demonstrated. It is a problem which should be of great importance to the designer of compressors and may well be equally important in low pressure turbine design. Although the predictions obtained using the model have only been compared with data obtained in a low Reynolds number turbine, the conclusions drawn should be equally applicable to high Reynolds number cases where transition is more likely to occur in favourable pressure gradients.

It is interesting to note here that when there are no disturbances in the inlet flow, laminar separation occurs in the linear cascade at about 80% \( s^* \). At this point on the turbine rotor suction surface, the predicted time-mean intermittency is about 80% and about 20% for both the 75% \( C_{x5} \) and 143% \( C_{x5} \) test cases respectively but there is no evidence of laminar separation in the data presented here or elsewhere. Results presented by Addison and Dong (1989) for a compressor blade suggest that separation does not occur in an intermittent boundary layer when the relaxation time between the passing of turbulent spots is insufficient for the boundary layer to revert from an attached turbulent to a separated laminar profile. It is also known that laminar separation can occur between the passing of spots at low intermittencies (Hart, 1985). However, the level of intermittency which is required to suppress the separation and whether intermittent separation is harmful to the aerodynamic performance are unknown. The effects of three-dimensionality and compressibility must also be investigated not only within this context but also in the wider context of attached flows.

Finally, the most important missing link in the whole problem is the prediction of the unsteady start of transition point \( t_H(t) \). The work of Addison and Hodson (1989) gave some indication of the important factors but is hardly adequate for design use.

**Conclusions**

This paper has presented a computational scheme for the calculation of transitional boundary layers in the presence of wake interactions. The scheme is based on a model developed from physical principles. It has been shown that the predicted development of a low Reynolds number turbine rotor suction surface boundary layer is in reasonable agreement with experimental observations. The model has also been used to explain observations made by other researchers at higher Reynolds numbers. Although reasonable agreement was obtained between predictions and measurements, it has been shown that further work is required to refine the model.

**References**


APPENDIX - THE RMS OF AN INTERMITTENT DISTRIBUTION

In the following analysis, a functional relationship between the intermittency \( \gamma \) of the boundary layer and the ensembled root-mean-square of the random unsteadiness as indicated by the surface-mounted hot-film anemometers is derived.

Consider a probability distribution \( p_2(x) \) with mean \( \mu_c \) and variance \( \sigma_c^2 \) which is a linear combination of two distributions \( p_a(x) \) and \( p_b(x) \) with means and variances \( (\mu_a, \sigma_a^2) \) and \( (\mu_b, \sigma_b^2) \) respectively.

Then

\[
p_c(x) = A p_a(x) + B p_b(x) \quad \text{where} \quad A + B = 1 \tag{A1}
\]

From the basic definition of the mean, we can write

\[
\mu_c = \int_\infty^{+\infty} p_c(x) \, dx = \int_\infty^{+\infty} (A p_a(x) + B p_b(x)) \, dx = A \mu_a + B \mu_b \tag{A2}
\]

From the definition of the variance, we can write

\[
\sigma_c^2 = \int_\infty^{+\infty} (x^2 - \mu_c^2) \, p_c(x) \, dx = \int_\infty^{+\infty} x^2 \, p_c(x) \, dx - [\mu_c^2] \tag{A3}
\]

since \( \mu_c \) is constant and \[ p_c(x) dx = 1 \]. Then, substituting for \( p_c(x) \)

\[
\sigma_c^2 = A \int_\infty^{+\infty} x^2 \, p_a(x) \, dx + B \int_\infty^{+\infty} x^2 \, p_b(x) \, dx - [\mu_c^2] \tag{A4}
\]

Using the definitions of variance for distributions \( A \) and \( B \), and substituting for \( \mu_c \) from (A1) then

\[
\sigma_c^2 = A [\sigma_a^2 + \mu_a^2] + B [\sigma_b^2 + \mu_b^2] - [A \mu_a + B \mu_b]^2 \tag{A5}
\]

where all of the terms are known.

For the case in question, we take \( A \) to be the laminar distribution and \( B \) the turbulent distribution and thus

\[
A = (1-\gamma) \quad \text{and} \quad B = \gamma \tag{A6}
\]

whence

\[
\sigma_c^2 = (1-\gamma) \sigma_a^2 + \gamma \sigma_b^2 + \gamma (1-\gamma) [\mu_a - \mu_b]^2 \tag{A7}
\]

and

\[
\mu_c = (1-\gamma) \mu_a + \gamma \mu_b \tag{A8}
\]