ROTORDYNAMIC FORCES DUE TO TURBINE TIP LEAKAGE - PART I: BLADE SCALE EFFECTS

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ABSTRACT
An experimental and theoretical investigation has been conducted on rotordynamic forces due to non-axisymmetric turbine tip leakage effects. This paper presents an actuator disc model which describes the flow response to a finite clearance at the rotor tip. The model simplifies the flow field by assuming that the radially uniform flow splits into two streams as it goes through the rotor. The stream associated with the tip clearance, or the undertumed stream, induces radially uniform unloading of the rest of the flow, called the bladed stream. Thus, a shear layer forms between the two streams. The fraction of each stream and the strength of shear layer between the two are found as functions of the turbine loading and flow parameters without resorting to empirical correlations. The results show that this model's efficiency predictions compare favorably with the experimental data and predictions from various correlations. A companion paper builds on this analysis to yield a model of the 3-D disturbances around an offset turbine and to predict the subsequent cross forces.

Nomenclature

<table>
<thead>
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<th>SYMBOL</th>
<th>DEFINITION</th>
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<tr>
<td>$B_2$</td>
<td>total enthalpy of meridional flow</td>
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<tr>
<td>$c$</td>
<td>absolute flow velocity; axial blade chord</td>
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<tr>
<td>$c_1$</td>
<td>lift coefficient</td>
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<tr>
<td>$D_{2R}$</td>
<td>turbine mean diameter</td>
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<td>$G$</td>
<td>factor defined in Eq. (40a)</td>
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<td>$H$</td>
<td>blade span</td>
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<td>$Q,q$</td>
<td>shear layer height</td>
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<td>$P$</td>
<td>pressure</td>
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<td>$R_s$</td>
<td>radius of rolled-up vortex</td>
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<td>$s$</td>
<td>blade pitch</td>
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<tr>
<td>$t$</td>
<td>radial tip clearance</td>
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<td>$U=\omega R$</td>
<td>turbine rotational speed at mean radius</td>
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$w$ | mean radius |
$w^+,w^-$ | relative velocity; work defect factor |
$Z_\alpha$ | Zweifel coefficient |
$\alpha$ | pitch angle; absolute flow angle |
$\beta = \frac{-\partial \eta}{\partial (1/H)}$ | efficiency loss factor |
$\beta$ | yaw angle; relative flow angle |
$\beta_m$ | mean rotor blade angle |
$\delta_j$ | clearance jet thickness |
$\Delta$ | thickness of undertumed layer downstream of the actuator disc |
$\phi = \frac{c_x}{U}$ | turbine flow coefficient |
$\Gamma$ | circulation of clearance vortex |
$\eta$ | turbine efficiency |
$\lambda$ | nondimensional undertumed flow rate |
$\pi$ | turbine pressure ratio; pi |
$\rho$ | density |
$\theta$ | azimuthal angle measured in the direction of rotation from the minimum gap location; underturning relative to the mean flow angle |
$\omega$ | angular velocity of rotor shaft rotation |
$\omega_y$ | vorticity of meridional flow |
$\Omega$ | angular velocity of rotor shaft whirl |
$\psi$ | meridional stream function |
$\psi = \frac{k_01 - k_03}{U^2}$ | turbine stage loading factor, or work coefficient |

Subscripts

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Superscripts
- the part of flow downstream which has crossed the bladed part of turbine flow
+ the part of flow downstream which was underturned due to the rotor tip gap

### 1. Introduction

One of the major problems which has always limited the development of high performance turbomachinery is the vibration of its structure. Ehrich and Childs (1984) have reviewed the various types of self-excited vibrations encountered in practice. Self-excited vibrations which occur in rotor systems are referred to as rotordynamic instability. The most publicized rotordynamic instability occurred during the development of the cryogenic turbopumps for the Space Shuttle Main Engine (SSME). One of the major causes was attributed to a phenomenon referred to in the US as the "Alford force". This phenomenon occurs due to fluid excitation forces caused by an asymmetric tip clearance distribution, feeding energy into the whirling motion of the shaft.

Thomas (1984) first identified the problem in high power steam turbines, and Alford (1965) identified the same problem in jet engines. They independently proposed an identical mechanism to explain the observed aerodynamic instability. Essentially, the local efficiency is higher in the smaller tip gap region, and the local torque and the tangential force exerted on the turbine by the fluid increase with the local efficiency. This assumption is based on empirical evidence that the efficiency of a turbine stage varies linearly with the tip clearance gap (Kofsky and Nusbaum, 1968). When integrated around the circumference, the net result is a force acting orthogonal to the displacement which adds energy to the forward whirling motion. Figure 1, from Ehrich & Childs, illustrates the mechanism.

Although other rotordynamic instabilities (e.g., destabilizing forces in labyrinth seals) have been extensively studied, relatively little effort has been expended on the tip clearance excitation force. Uriachs (1983) and Vance & Laudadio (1984) established the existence and linearity of the Alford force in an unshrouded turbine and an axial fan, respectively. However, a basic physical understanding of the influence of various geometric and flow related parameters in the generation and scaling of the tip clearance excitation force was still lacking.

Therefore, the objective of this investigation was to gain physical understanding of the force generation mechanism via experimental and theoretical methods. The current investigation's experimental methods and results were previously reported by Martinez-Sanchez et al. [7]. This paper presents the first part of a model developed during the theoretical phase of the investigation.

### 2. Model Description

The flow response to an eccentric turbine involves the three scales of the tip gap, $t$, the blade span, $H_b$, and the turbine radius, $R$. They are typically in ratios of the order $t/H_b \approx 0.01$ and $H_b/R \approx 0.1-0.3$ (Figure 2). Therefore, the gap-scale effects such as the leakage flow velocity, which is driven by local tip conditions, can be decoupled from the other two larger length scales. The larger length scale effects such as the tip blade loading become boundary conditions. The blade-scale effects include the radial migration of through flow towards the tip gap and the underturn of flow in the outer region of the blade span due to the gap leakage flow. These effects are influenced by not only the tip gap effects, which determine the leakage rate, but also the radius scale effects, which determine the local turbine inlet and outlet conditions. In reality, the inlet and outlet conditions would vary azimuthally for an eccentric turbine. However, at the blade scale, these boundary conditions can be assumed to be axisymmetric at the local value. Thus, the blade scale effects link the effects of the tip gap and the turbine radius. Finally, at the turbine radius scale, the turbine eccentricity causes azimuthal

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variations of flow variables (e.g. the flow coefficient). The effects of two smaller scales are seen mainly as connecting conditions between the upstream and downstream flows.

The differences in the relative importance of unsteady effects are also pronounced at different scales. A rotor simultaneously spinning at angular frequency, \( \omega \), and whirling at angular frequency, \( \Omega \), can have a radius scale reduced frequency, \( \Omega R/c_\tau \), close to the order of unity. However, at the two smaller scales, the reduced frequencies, \( \Omega h/c_\tau \) and \( \Omega L/c_\tau \), are orders of magnitude smaller than unity. Therefore, the unsteady effects need to be considered only at the radius scale, and the flow can be assumed to be quasi-steady at the other two smaller scales.

For the case of a turbine with a whirling rotor, both the stator row and the rotor blade rows are unwrapped and collapsed into a single actuator disc (Figure 3). In this coordinate system, \( x \) is the direction of the through flow, \( y \) is the tangential (azimuthal) direction, and \( z \) is the radial direction.

The analysis proceeds at two levels, as shown in Figure 3. The first is a quasi-steady blade scale analysis in the meridional (xz) plane. This analysis focuses on the radial flow redistribution effects due to the existence of rotor tip gap and does not consider the tangential flow redistribution effects due to a whirling turbine. This more global, radius scale (xy) analysis, which occurs upstream and downstream of the disc. The second is an unsteady radius scale analysis in the radial (xy) plane which focuses on the tangential flow redistribution effects due to a whirling turbine. This more global, radius scale (xy) analysis, presented in a companion paper, utilizes the results from the blade scale (xz) analysis as connecting conditions across the actuator disc. Such a framework can accommodate a flexible, modular approach in which various submodels at different length scales can be separately incorporated into the analysis.

This paper exclusively presents the blade scale (xz) model. It is capable of analytically predicting the effects of turbine rotor tip clearance on the losses and the turbine performance. The blade scale model builds on previous efforts by Gauthier (1990) and Martinez-Sanchez and Gauthier (1990).

The following is assumed in deriving the governing equations:

1. Incompressible, inviscid flow.
2. Turbine stage collapsed in the axial direction to \( x=0 \).
3. The blades guide the flow perfectly (except for the leakage flow) so that the relative flow exit angle is same as the blade exit angle.
4. The flow is axisymmetric \( (\partial/\partial y = 0) \).
5. Except for the rotor tip gap, the blade geometry is assumed to be radially uniform \( (\partial/\partial z = 0) \) and to be equivalent to that at the mean radius.
6. Flow conditions are radially uniform at the stator exit.

The blade actuator disc consists of a full span stator row and a partial span rotor row as shown in Figure 4. Far upstream of the stator row is referred to as \( 0 \). Inlet to the stator row is referred to as station 1 and the stator exit is called \( 2s \). Inlet to the rotor is referred to as \( 2r \) and rotor exit is referred to as \( 3 \). Far downstream of the rotor row on the blade scale is called station 4.

Figure 5 shows the turbine blading geometry with the velocity triangles. \( U \) is the turbine rotational speed; \( c \) is the absolute flow velocity; and \( w \) is the relative flow velocity. Angles \( \alpha \) and \( \beta \) refer, respectively, to the absolute and the relative flow angles.

3. Tip Scale Analysis

Martinez-Sanchez and Gauthier (1990) showed that, because of the leakage flow roll-up, an underturned layer can be identified downstream of the rotor, containing equal amounts of leakage and blade-region flow. The fluid in this region has undergone less turning than the main blade flow, but not zero turning, and has therefore done a finite amount of work. This underturned fluid is dealt with in this section.

The blade tip region has been analyzed using a variety of approaches. The simple model of Rains (1954), which is most appropriate for thin, lightly loaded blades, uses ideal, pressure-driven flow concepts to derive the velocity of the gap 'jet'. Even for the case of the thicker turbine blading, ideal flow is a fairly good approximation. For example, Rains] gave a criterion for viscous forces to be negligible, and many turbines satisfy this condition. On the other hand, although the effects of chordwise pressure gradients on thick blade tip flows, as well as that of relative wall motion are still potentially significant, they have not been modeled.

The gap jet is known to interact strongly with the passage flow and to roll up into a concentrated vortex-like structure. Rains derived a semi-empirical expression for its trajectory, Lakshminarayana (1970) also used empirical information on the tip vortex location.
and strength to predict details of the blade pressure distribution. In fact, the strength of the vortex was explicitly related to a "partial blade tip loading parameter", K, varying from 0 to 1, and inferred from extrapolation of surface pressure measurements near the tip to the end wall.

This section introduces an analytical approach which leads to simple, but accurate expressions for the location and size of the leakage vortex. This can then be used in calculating the flow leaving angle of, and hence the work done by the leakage flow.

Figure 6 shows schematically the essential features of the leakage flow. The fluid approaches a blade (here represented as a flat plate) with a relative velocity, \( \vec{W}_2 \), which evolves into the passage flow velocity, \( \vec{W}_{pass} \), at locations away from the tip gap. Under the action of the pressure differential across the blade, a jet of leakage flow at velocity \( \vec{W}_{jet} \) escapes under the blade. This jet penetrates a certain distance into the passage, but is eventually stopped by the main flow, which separates the jet from the wall, turns it backwards, and leads to the formation of a rolled-up structure containing both leakage and passage fluid. This "collision" of the two streams is again shown in Figure 7 in plan form, and Figure 8 shows a schematic of the flow structure seen in a cut such as a-a in Figure 7 with leakage fluid shown dashed.

Consider the situation at points along the jet separation line, such as P in Figures 7 and 8. Ignoring frictional effects, the two streams which meet there (jet and passage flows) can both be traced back, along different paths, to the inlet flow, and hence have equal total pressures and temperatures. Since they also have equal static pressures along their contact line, (and generally similar static pressures throughout the region), these two streams must have equal velocity magnitudes. If the section a-a is perpendicular to OP, we can think of point P (Figure 8) as the common stagnation point of the two colliding flows containing both leakage and passage fluid. This "collision" of the two streams is again shown in Figure 7 in plan form, and Figure 8 shows a schematic of the flow structure seen in a cut such as a-a in Figure 7 with leakage fluid shown dashed.

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Notice that the transverse momentum balance of a fluid element near point P requires that both transverse colliding flows must bring equal and opposite momentum fluxes to the rolled-up structure. Since the two velocities are equal, we find that equal mass flows must be entering the rolled-up structure from both fluids. In other words, the clear and dashed areas in Figure 8 must occupy equal fractions of the total "vortex" cross section. Let \( \delta_{jet} \) be the jet thickness, and \( w_{||}, w_{\perp} \) the common components along and across OP of the colliding streams. The rate of increase of the cross-section \( A_{\perp} \) of the rolled structure along OP is then given by

\[
\frac{dA_{\perp}}{ds} = 2w_{\perp} \delta_{jet} \tag{1}
\]

or, calling \( \theta = \tan^{-1} \frac{w_{\perp}}{w_{||}} \), i.e., the angle made by the separation line OP and the blade itself,

\[
\frac{dA_{\perp}}{ds} = 2 \delta_{jet} \tan \theta \tag{2}
\]

where s is measured along the vortex trajectory.

The precise shape of the rolled-up structure is more difficult to establish, but it seems reasonable to model it as a (half) cylindrical ideal vortex in a cross-flow. Following Batchelor (1967) such a vortex is described by the stream function

\[
\Psi = 1.298 w_{||} R_J \left( \frac{3.83}{R_D} \right) \sin \theta \tag{3}
\]

where \( R_D \) is the radius of the dividing streamline, \( J_1(x) \) is the Bessel's function of the first order (with a zero at \( x=3.83 \)) and \( (r, \theta_1) \) are polar coordinates. The vorticity in this flow is distributed inside the semi-circle of radius \( R_D \) in proportion to \( \Psi \):

\[
\xi = \left( \frac{3.83}{R_D} \right)^2 \Psi \tag{4}
\]

and is zero outside. Integration of \( \xi \) gives an overall circulation \( \Gamma = 6.83 w_{||} R_D \), whereas integration of \( r \sin \theta_1 \) gives a center of vorticity height of \( z_C = 0.460 R_D \). We thus make
\[ A_1 = \frac{1}{2}\pi R_D^2, \text{ and measuring distance along the} \]
\[ \text{blade (}x_{bl}=s \cos \theta), \text{ we can integrate Eq. (2) to} \]
\[ R_D = \sqrt{\frac{4(\tan \theta)}{\pi (\cos \theta)}} \delta_{jet} x_{bl} \]  

(5)

The trajectory of the vortex center then follows (Figure 9) as

\[ y_c = x_{bl} \tan \theta - \frac{R_D}{\cos \theta} \]  

(6)

To complete the analysis, the angle \( \theta \) must now be determined. From our discussion of the separation line OP, this angle was shown to be half of the angle \( \beta \) between the blade and the jet flow, i.e., \( \theta = \beta/2 \) (Figure 9). This angle \( \beta \) follows from the simple local analysis first proposed by Rains (1954) which applies to thin blades when viscous effects can be neglected. In Figure 10, \( w_p \) and \( w_s \) are the flow velocities on the pressure and suction sides of the blade, respectively. Application of Bernoulli's equation relates these velocities to the corresponding pressures:

\[ w_p = \sqrt{w_p^2 - 2 \frac{p_p - p_2}{\rho}} \]  

(7)

\[ w_s = \sqrt{w_s^2 + 2 \frac{p_p - p_2}{\rho}} \]  

where \( p_2, w_2 \) correspond to inlet conditions. On the other hand, the leakage jet emerges from the gap with a velocity component perpendicular to the blade of

\[ w_G = \sqrt{2 \frac{p_p - p_s}{\rho}} \]  

(9)

and its component parallel to the blade is simply \( w_p \), since no momentum is added or lost in that direction during passage through the gap. It can be verified that the net magnitude \( w_{jet} \) of the jet velocity is then equal to \( w_s \), as indicated previously. We then obtain (Figure 10)

\[ \tan \beta = \frac{w_G}{w_p} = \sqrt{\frac{(c_p)_p - (c_p)_s}{1 - (c_p)_p}} \]  

(10)

where \( c_p = 2(p - p_2)/\rho w_2^2 \) in each case. Note that \( (c_p)_p - (c_p)_s \) is the local lift coefficient, \( c_i \), referred to the relative turbine inlet velocity. Using the half-angle trigonometric formulae,

\[ \tan \theta = \frac{\sqrt{(c_p)_p - (c_p)_s}}{\sqrt{1 - (c_p)_s} + \sqrt{1 - (c_p)_p}} \]  

(11)

Notice that, as shown in Figure 9, the vorticity vector corresponding to the shear between the jet and the adjacent passage flow is inclined at \( \theta \) with respect to the blade, i.e. it is parallel to the outer edge OP (Figure 7) of the rolled-up structure. This is also the direction of the mean flow between the two sides of the shear layer, which means that the shear vorticity is not convected at all towards the line OP. The only reasons the vorticity \( \Gamma \) rolled up into the structure increases with downstream distance is that the growth of \( R_0 \) gradually overlaps more and more of the shear vorticity. In this sense, the commonly invoked view of the rolled-up vortex growing by the convection of shed vortices must be used with caution.

Eqs. (5), (6), and (11) can now be used to calculate the vortex geometry if the suction and pressure side \( c_p \) distributions are prescribed. A simple approximation can be obtained using the theory of lightly loaded thin wing profiles. In this approximation, \( (w_p + w_s)/2 \equiv w_s \), which when used in Eqs. (7, 8) reduces both \((c_p)_p\) and \((c_p)_s\) to functions of \( c_i = (c_p)_p - (c_p)_s \) alone. Using this in Eq. (2) gives finally

\[ \theta = \tan^{-1} \left( \frac{c_i}{4} \right) \]  

(12)

where \( c_i \) is the lift coefficient per unit blade span and is related to the Zweifel coefficient, a blade loading parameter. Zweifel coefficient, \( Zw \), and the lift coefficient, \( c_i \), are determined as follows:
where \( s/c \) is the blade pitch normalized by the axial blade chord, and

\[
c_i = \frac{ZW \cos^2 \beta_2}{\cos \gamma \cos \beta_m} \tag{14}
\]

where \( \gamma \) is the rotor blade stagger angle.

4. Comparison to Vorticity Dynamics Model and to Data

Chen's similarity analysis has provided a means of correlating a variety of rolled-up vortex data. Transverse distances are normalized by gap width, \( t \), and axial distance (or time-of-flight) are characterized by a parameter

\[
t^* = \frac{x}{c_i} \left( \frac{\Delta P}{\rho} \right)^{1/2} \tag{15}
\]

where \( x \) and \( c_x \) are the axial distance and velocity and \( \Delta P = p_p - p_i \). The data from many experiments (mainly from compressor cascades) correlate well with \( t^* \). In addition, a calculational method was developed by Chen to track a series of shed tip vortices from an impulsively started plate, which represents the situation seen from a convective frame as the flow passes over a blade. The calculated results were shown to also correlate well with \( t^* \) and with the data. We use the correspondences

\[
\frac{c_x}{w_2} = \cos \beta_2, \quad \frac{x}{x_m} = \cos \beta_m \quad \text{where} \quad \beta_2 \quad \text{and} \quad \beta_m
\]

represent the inlet and average relative flow angles at the rotor, respectively, to derive

\[
\frac{x_{bl}}{\delta_{m2} / t} = \frac{\sqrt{2}}{c_d} \frac{w_2 \cos \beta_2}{\cos \beta_m} t^*. \tag{16}
\]

where \( c_d = \delta_{m2} / t \) is the gap discharge coefficient. Note also that \( \frac{w_2}{w_G} = 1 / \sqrt{c_l} \).

For an approximate comparison, Rains' (1954) values \( c_d = 0.785, c_i = 1.35 \)

\( \cos \beta_2 = 1.1 \) are used to relate \( t^* \) to \( x_{bl} \), and then calculate the vortex trajectory using Eqs. (5), (6), (11) and (12). The results are compared in Figure 11 to those reported in Chen. The agreement with the data is satisfactory. Additional verification against the theory of Chen is provided by comparing the predictions of both theories regarding the "center of vorticity" location in a cross-plane similar to that shown in Figure 8. In order to be consistent with the Chen's calculations, we have included here both the rolled-up vorticity \( \Gamma = 6.83 w_1 R_D \) and a vorticity \( 2 w_1 \) per unit length (perpendicular to \( \xi \) of the not-yet-rolled shear layer. In calculating the distance \( z_c \) between the center of vorticity and the wall, we took this latter contribution to be at a distance \( \delta_{m2} \), and that of the rolled-up vortex to be at \( \delta_{m2} + 0.46 R_D \). The results are shown in Figure 12, which again shows good agreement between our method and that of Chen.

5. Blade Scale Analysis

This section focuses on the rest of the flow, which goes through the blades as designed. The azimuthal momentum equation for the flow is

\[
\bar{c}_1 \cdot \nabla c_y = 0 \tag{17}
\]

where \( \bar{c}_1 \) is the meridional velocity defined as

\[
\bar{c}_1 = \bar{c}_x + \bar{c}_z
\]

The vorticity equation also reduces to

\[
\bar{c}_1 \cdot \nabla \omega_y = 0 \quad (\omega_y = \frac{\partial c_z}{\partial z} - \frac{\partial c_x}{\partial x}) \tag{18}
\]

Also, the Bernoulli equation reduces to

\[
\bar{c}_1 \cdot \nabla B_1 = 0 \quad B_1 = \frac{p + \frac{1}{2} c_1^2}{\rho} \quad (\frac{1}{2} c_1^2 + \frac{1}{2} c_2^2) \tag{19}
\]

Continuity is ensured by introducing the stream function \( \psi(x, z) \) for the meridional flow where

\[
c_z = \frac{\partial \psi}{\partial z}, \quad c_x = -\frac{\partial \psi}{\partial x}, \quad \text{so that}
\]

\[
\omega_y(\psi) = \nabla_1^2 \psi \quad (\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}) \tag{20}
\]
Thus, the meridional flow \((c_x, c_z)\) is decoupled from \(c_y\), and, therefore, can be solved for first.

Upstream of the stage \((x < 0)\), the flow is assumed to be irrotational \((\omega_y = 0)\). Thus, \(\Psi\) obeys Laplace's equation.

Gauthier (1990) showed that it is a good approximation to assume that the axial velocity at the rotor is piece-wise constant, with a discontinuity at the blade tip. This implies constant work extraction in each of the two regions (blade and gap), and so the \(\omega_y\) vorticity source is concentrated on a surface at their common boundary.

Downstream \((x > 0)\), the flow is radially non-uniform, and may show a contact discontinuity on the tip streamlines, thus dividing the flow into two streams, the underturned flow due to the rotor gap (marked by a superscript +) and the main passage flow (marked by a superscript -). The presence of the rotor tip clearance induces a radially non-uniform work extraction, which, in turn, gives rise to a radially non-uniform Bernoulli constant downstream of the rotor. Because of the relationship

\[
\omega_y = \frac{dB_1}{d\psi}
\]

this creates \(\omega_y\) vorticity, which is then convected along each streamline. Then by Eq. (20), \(\Psi\) obeys Poisson's equation for \(x > 0\).

From the definition of \(B_1\), with the continuity constraint and the assumption of spanwise uniform blading,

\[
B_{x1} - B_{x3} = \frac{p_1 - p_3}{\rho}
\]

Since \(dB_{x1}/d\psi = 0\) by the assumption of irrotationality, Eq. (21) can be rewritten as

\[
\omega_{y3} = -\frac{d(B_{x1} - B_{x3})}{d\psi} = -\frac{d}{d\psi}\left(\frac{p_1 - p_3}{\rho}\right)
\]

Thus, the radial distribution of static pressure drop can be used to determine the downstream vorticity.

For the bladed stream, \((p_1 - p_3)/\rho\) is equivalent to the stagnation enthalpy drop, given by Euler's turbine equation, minus the kinetic energy gain. According to Euler's equation, the stagnation enthalpy drop, \(-\Delta h_i\), is given as

\[
-\Delta h_i = U[c_{z2r} - c_{z3}]
\]

where \(U\) is the turbine's rotational speed. The additional subscript \(r\) in \(c_{z2r}\) denotes the rotor end of the axial space between stator and rotor. The stator exit end would, in general, have a different tangential velocity, \(c_{z2s}\). The kinetic energy gain, DKE, is \(\Delta K.E. = \frac{1}{2}(c_{z3})^2\). At the rotor exit \((3)\), the flow has split into two streams. For the bladed stream,

\[
c_{z3} = U - c_{z3} \tan \beta_3
\]

Thus,

\[
\left(\frac{p_1 - p_3}{\rho}\right) = Uc_{z0} \tan \alpha_2 - \frac{1}{2}
\]

\[
\left[U^2 - (c_{z3})^2 \tan^2 \beta_3\right]
\]

For the underturned stream, \(c_{z2}\), is the same as that for the bladed stream because the flow is assumed to be radially uniform upstream of the rotor. Thus,

\[
\left(\frac{p_1 - p_3}{\rho}\right) = U\left[c_{z0} \tan \alpha_2 - \left(U - c_{z3}\right)\right] + \frac{1}{2}\left(c_{z3} \tan \beta_3\right)^2
\]

\[
\frac{1}{2}(c_{z3})^2 + \frac{1}{2}\left(c_{z3} \sin \theta \cos \beta_m\right)
\]

The last term in Eq. (27) is included to account for the kinetic energy dissipated when the flow which leaked through the rotor gap collides with an equal amount of the passage flow before rolling up into vortices.

\(d/d\psi\) in Eq. (23) can be expressed as

\[
1\text{This inter-blade gap effect is modeled and explained in Section 3.4 of the companion paper.}
\]
\[
\frac{d}{d\psi} = \left( \frac{1}{c_x \partial z} \frac{\partial}{\partial c_x} \right)_{z=0+} \frac{\partial}{\partial c_x}
\]  
(28)

Then from Eqs. (23), (26), (27), (28), the equation for \( \psi \) becomes

\text{Upstream (}x < 0): \quad \nabla_\perp^2 \psi = 0 \tag{29}

\text{Downstream (}x > 0): \quad \nabla_\perp^2 \psi = Q \delta \left( \psi - \psi_{tip} \right) \tag{30}

where \( Q = \int_0^\infty \omega_y dz = B_{13}^+ - B_{13}^- \)

\( \delta \) is Dirac's delta function, and \( Q \) is the strength of the \( y \) component of the shear layer between the underturned and bladed streams. From Eqs. (22), (26), and (27) and the fact that \( B_{13} \) is continuous,

\[
Q = U(c_{y3}^- - c_{y3}^+) - U(c_{y3}^+ - c_{y3}^-) + \frac{1}{2} \left[ (c_{y3}^-)^2 - (c_{y3}^+)^2 - \left( c_{y3}^+ \frac{\sin \theta}{\cos \beta_m} \right)^2 \right] \tag{31}
\]

can be obtained. The boundary conditions are

\( \psi(x,0) = 0 \)
\( \psi(x,H) = c_{xH} \)
\( \psi(x = 0-, z) = c_{xH}z \)
\( \frac{\partial \psi}{\partial \xi} (x = 0+, z) = 0 \tag{32} \)
\( \psi_1(z) = \psi_3(z) \)
\( \frac{\partial \psi_1(z)}{\partial z} = \frac{\partial \psi_3(z)}{\partial z} \)

The axial velocities for each stream are related to the shear parameter, \( Q \), using overall continuity, as

\[
c_{z3} = c_{z2r} \left( 1 - \frac{q}{2} \right) \tag{33}
\]

\[
c_{z3} = c_{z2r} \left( 1 - \frac{q}{2} \right) \tag{34}
\]

where \( q = Q/c_{z2r}^2 \) and \( I \) is the amount of underturned flow normalized by the total through flow.

The tangential velocity for the underturned layer is given by

\[
c_{y3}^+ = U - c_{x3}^+ \frac{\cos \theta \sin(\beta_m - \theta)}{\cos \beta_m} \tag{35}
\]

and that for the bladed stream is given by

\[
c_{y3}^- = U - c_{x3}^- \tan \beta_3 \tag{36}
\]

One feature of the actuator disc approximation is that only half of the total change visible far downstream of the disc occurs at the disc with the other half occurring in the flow downstream from the disc [14]. Therefore, far downstream on the blade scale, (at \( x \equiv 0^+ \)), the axial velocities are as follows

\[
c_{x4}^+ = c_{x2r} \left( 1 + q(1 - \lambda) \right) \tag{37}
\]

\[
c_{x4}^- = c_{x2r} \left( 1 - \lambda q \right) \tag{38}
\]

Since the shear between the two layers is simply convected downstream, the tangential velocities far downstream remain the same as those at the disc which are given by Eqs. (35) and (36).

The underturned flow consists of the flow which leaked through the rotor gap and an equal amount of the bladed flow entrained by the tip vortex (Figure 8). Therefore, the amount of flow leaked through the gap is \( \lambda/2 \), which is shown (Gauthier, 1990) to be a function of the tip gap height, \( t/H \), and \( \lambda \),

\[
\frac{1}{H} = \frac{\lambda}{2} \left[ 1 - \left( 1 - \frac{\lambda}{2} \right)^2 \right] \tag{39}
\]

In turn, \( q \) is a function of \( \lambda \) and the geometry of the turbine blading. Substituting Eqs. (33)–(36) into Eq. (31) results in a quadratic equation for \( q \).
\[
\begin{align*}
[G(1 - \lambda)^2 - (\lambda \tan \beta_3)^2] \left( \frac{q}{2} \right)^2 + \\
2 \left[ 2.0 + G(1 - \lambda) + \lambda \tan^2 \beta_3 \right] \left( \frac{q}{2} \right) + \\
\left[ G - \tan^2 \beta_3 \right] = 0
\end{align*}
\]

where

\[
G = \left( \frac{\cos \theta \sin (\beta_m - \theta)}{\cos \beta_m} \right)^2 + \left( \frac{\sin \theta}{\cos \beta_m} \right)^2
\]

Thus, for a given \( t/H \), the values of \( \lambda \) and \( q \) which satisfy the system of two Eqs. (39) and (40) can be found. Once \( \lambda \) and \( q \) are determined, all of the velocities can be determined from Eqs. (33)-(36).

Furthermore, other far downstream (4) conditions such as the thickness of the underturned layer (\( \lambda \)) and the pressure are needed as connecting conditions at the radius scale. From continuity relations between Stations 3 and 4, the following expression for \( \Delta H \) is obtained.

\[
\Delta H = \frac{\lambda}{H} \left( 1 - \left( \frac{1 - \lambda}{2} \right) \frac{q}{2} \right) \frac{1 + q(1 - \lambda)}{1 + q(1 - \lambda)}
\]

Also, keeping in mind the assumption of lack of radial redistribution of flow in the stator and the assumption of quasi-steady flow on the blade scale which permits the use of Bernoulli equation, the expression for pressure drop between stations 0 and 4 (or \( x=0^- \) and \( x=0^+ \)) can be obtained as follows:

\[
\begin{align*}
\frac{P_0 - P_4}{\rho U^2} &= \frac{c_{x2} \tan \alpha_2 - \frac{1}{2} + \frac{1}{2} \left( \tan \beta_3 \frac{c_{x3}}{U} \right)^2}{U} \\
&= \frac{1}{2} \left( \frac{c_{x3}}{U} \right)^2 + \frac{1}{2} \left( \frac{c_{x4}}{U} \right)^2
\end{align*}
\]

5.1 Work Defect and Efficiency Loss Determination

Application of the Euler equation to both fluids gives the work done per unit mass by each stream:

\[
W^* = U \left( c_{x2} \tan \alpha_2 - c_{x3} \right)
\]

\[
W^* = U \left( c_{x2} \tan \alpha_2 - c_{x3}^* \right)
\]

Then, the total turbine work per unit mass is

\[
W^- = U \left( c_{x2} \tan \alpha_2 - c_{x3}^- \right)
\]

In coefficient form, the power extracted by the turbine, and hence, the tip loss coefficient can also be calculated easily.

\[
\Psi = \frac{1}{mU^2} \left[ \int_0^{\infty} \left( h_1 - h_3 \right) \rho d\psi + \int_{\infty}^{1} \left( h_1 - h_3 \right) \rho d\psi \right]
\]

The total pressure drop in the mixed out region is given by

\[
\frac{P_0 - P_4}{\rho} = \frac{P_0 - P_4}{\rho} - \frac{1}{2} \frac{c_{x4}^2}{\rho}
\]

where \( P_4 \) is at far downstream of the disc (on the blade scale), and we have taken advantage of axial and radial velocities being zero at this location. The static pressure drop between Stations 0 and 4 is given by Eq. (42). Then, the efficiency is

\[
\eta = \frac{\Psi}{\left( \frac{P_0 - P_4}{\rho U^2} \right)}
\]

where \( \Psi \) is as given by Eq. (46). The efficiency loss factor follows as

\[
\beta = \frac{1 - \eta}{t/H}
\]

The work defect factor (different from \( \beta \)) is defined as

\[
w = \frac{\Psi - \Psi_0}{\Psi_0 \frac{t/H}{t/H}}
\]
where $\Psi_0$ is the work coefficient for zero leakage ($\lambda = 0$).

5.2 Model Predictions

We can now compare the calculated losses to those reported in the experimental literature. We rely for this on the compilation by (1986) which gives data for ten cases (nine different turbines) over a wide range of parameters. Waterman reports for each case the tip values of the work coefficient, $\Psi$, degree of reaction, $R$, flow coefficient, $\phi$ and individual blade loading, Zweite! coefficient.

One potential difficulty in application is that only tip parameters are given, whereas from the nature of our theory we suspect that mean parameters might be more appropriate.

Scanning Table 1, we first notice a large disagreement for Case 1 (Koltskey turbine). This is an impulse rocket turbopump stage with an extremely large tip loading ($\psi = 7.0$). Case 4, with a very high reaction, is also substantially under-predicted, which may point out an insufficient predicted underturning angle $\phi$ for these conditions. The rest of the cases are well predicted. Excluding Case 1, the mean squared error is $E^2 = 0.1162$ and the mean error is $E = -0.1408$. If Case 4 is removed, the two quantities would be, respectively, 0.0363 and -0.0498. Perhaps more effort should be devoted to an understanding of the leakage and underturning for high reaction rotors.

6. Summary and Conclusions

A model has been developed to illuminate the effects of spanwise flow redistribution caused by the presence of a small rotor blade-tip gap. To this end, the blade to blade details are ignored by using an incomplete actuator disk formulation which collapses both stator and rotor to a plane, across which connecting conditions are imposed.

The model assumes that near the gap region, the underturned layer originates partly from the gap flow and partly from the passage flow, both leaving the passage in the form of rolled-up tip vortex. The trajectory and other details of this vortex are calculated using a simple model involving the collision of the ideal pressure-driven leakage jet with the passage fluid. This model was calibrated against both data and theory of Chen [13]. The new theory allows prediction of the vortex strength and trajectory.

The model predictions were then compared to a set of data involving nine different turbines. With the exception of one anomalous case, the calculated efficiency loss factors are reasonably close to the data, showing less deviation than the loss correlations of Ainley, Soderberg, Roeleke, Koltskey, and Lakshminarayana.

These results suggest that upstream flow redistributions, which have been largely ignored so far, may be of importance in understanding the basic physics of tip leakage effects.

6. Acknowledgements

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7. Bibliography


### Table 1: Efficiency loss and work defect factors (calculated from the blade scale analysis) compared to data. The last line is computed with modified work coefficient chosen for near-exit flow.

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<th>Case</th>
<th>Author</th>
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<th>f</th>
<th>Yo</th>
<th>R</th>
<th>(t/H_b)</th>
<th>b_{data}</th>
<th>b_{calc}</th>
<th>w_{calc}</th>
<th>K_{calc}</th>
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<td>0.35</td>
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**Figure 1:** Thomas-Alford destabilizing mechanism
Figure 2: Flow features at different length scales in a turbine with an eccentric rotor.

Figure 3: Modeling the flow field...

Figure 4: A blade scale view of the actuator disc, showing various axial stations.

Figure 5: Turbine velo.
Figure 6: Schematic of the colliding leakage jet and passage flows.

Figure 7: Planform view of Figure 6

Figure 8: Flow pattern in plane a-a of Figure 7.

Figure 9: Position and width of rolled-up leakage vortex

Figure 10: Velocities and angles associated with roll-up
Figure 11: Trajectory of vortex centroid compared to data and theory of Chen (1991)

Figure 12: Coordinates of vorticity centroid for tip clearance vortex