On The Practical Stability and Its Application to Nonlinear Control of Surge

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ABSTRACT

This paper first proposes a new control strategy, stabilization on the sense of practical stability, to actively stabilize the axial compression systems regardless their instability modes. Then, the theory of practical stability is applied to analyze the practical stability of the pure surge system which is a compression system without any nonaxisymmetric disturbances that grow into rotating stall. This analysis reveals that the upper limit of the amplitude in which the system surges is determined by the nonlinearity of the compressor characteristic. Thus, a nonlinear controller is designed to control the trajectory of the pure surge system by modifying the nonlinear terms in the expression for the compressor characteristic. Numerical simulation shows that the nonlinear controller can effectively shrink the size of the trajectory of the pure surge system, and therefore stabilize the system in the sense of practical stability.

1. INTRODUCTION

One of the most important purposes of the analysis of linear and nonlinear dynamics of axial compression systems is to provide a “good enough” strategy for use in active stabilization of the systems. Ideally, we would like to be able to accurately predict the onset of the compressor instability and clearly understand which factors are important and how they affect the compressor performance. The common criterion derived from linear stability analyses states that rotating stall inception will occur at the peak (zero slope point) of the compressor characteristic. Although this criterion appears to furnish a rough “rule of thumb”, the counter examples in which it does not hold can also be readily found. Greitzer (1981) provided several such examples. In other words, the stability boundary for a certain compression system may or may not coincide with the linear stability limit. Experiments show that, in most cases, such a practical stability boundary is lower than the linear stability boundary, namely that a linearly stable operating point could be nonlinearly unstable. Moreover, the actual operating point is set even lower than the practical stability boundary because a sufficient margin is required such that the system can absorb all of the arbitrary initial disturbances.

In order to improve the performance of the compressor, the control strategy suggested by Epstein et al. (1989) is to extend the linear stability boundary to a previously unstable region by assuming that the practical stability boundary can be extended correspondingly. Paduano (1991) and Haynes et al. (1993) implemented this idea to a single-stage compressor and a three-stage compressor respectively.

In recent years, the nonlinear dynamics of axial compression systems is investigated extensively by many researchers (Adomaitis et al. 1993, Badmus et al. 1993, McCaughan 1994, Lin 1995). In spite of a variety of different approaches, the common point is that linearization of the nonlinear model of the compression system is not a good approximation when the compressor operates at the region close to the peak of its performance curve where the nonlinear dynamics of the compression system dominates. This implies that a successful controller must be designed under the consideration of global dynamics of the system rather than the local behavior in a small neighborhood of the desired state of the system. A large number of nonlinear control schemes have been developed. For the pure surge system where the only instability mode is surge, Badmus et al. (1995) applied the feedback linearization theory to the sys-

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tem and experimentally compared their new controller with a linear controller. The closed loop performance achieved by the nonlinear controller is shown to undergo relatively less transient excursions. In case of the rotating stall is the only instability mode, the control strategy is to eliminate the hysteresis loop with respect to rotating stall so that the domain of attraction of the equilibrium point can be extended to the whole phase space (Adanaitis et al., 1993, Badmus et al. 1993, Sepulchre & Kokotovic, 1996). The validation of such a control strategy has experimentally been demonstrated by Behnken et al. (1995) and other researches. One of the advantages of these nonlinear controllers is that they require relatively lower number of sensors and actuators, relatively lower bandwidth of the actuators than the linear controllers. However, one argument against this control strategy is that one may need to predict whether the instability mode is rotating stall or surge before he turned on the associated controller, while such a prediction is in general not available in reality. As shown experimentally by Greitzer (1976) and theoretically by McCaughan (1990), the compression systems exhibit complicated instability modes. For instance, in addition to pure surge and hysteresis loop with respect to pure rotating stall, the nonlinear phenomena observed in the compression system include classical surge, hysteresis loop with respect to surge. Moreover, Day et al. (1978) observed in his experiment that at a low design flow coefficient rotating stall occurs without any hysteresis associated with it. Lin and McCaughan (1996) theoretically demonstrated that there is another possible instability mode in the one-mode truncated system, which cannot be described in terms of the regular stability concept. That is, when parameter \( S \) and the Greitzer's \( B \) parameter are large enough, the compression system surges over one cycle under the influence of a very small initial nonaxisymmetric disturbance, and then returns to its design operating point after that surge cycle. The system is considered to be unstable because a compressor will be in danger if it surges over one cycle frequently and randomly. This instability mode is referred to as "surge-like" instability.

The origin of applying the concept of practical stability to the compression system is following: Experiments show that the compressor always lose its stability to the nonaxisymmetric disturbance first, even though the system eventually surges. Hence, it is suggested that if the initial nonaxisymmetric disturbance can be suppressed the system can be stabilized regardless its final instability modes. This implies that it may not be necessary to extend the domain of attraction of the linearly stable points to the entire phase space. The performance of a compression system would be satisfactory if its trajectories starting inside a region that contains all possible initial disturbances would not go outside a neighborhood of the operating point whose size is determined by practical satisfaction of the performance. This is exactly the meaning of practical stability as defined in section 3.

Further, it is realized that, when the operating point is linearly stable but practically unstable, it is the nonlinear interaction between the mean flow and the nonaxisymmetric disturbance that make the disturbance grow and push the system away from its linearly stable operating point. Lin and McCaughan (1990) studied this nonlinear interaction and concluded that as long as the linearized system is stable, the initial nonaxisymmetric disturbance can be suppressed by sensing and controlling the mean velocity only. For convenience, the analysis is briefly reviewed here. If the Moore-Greitzer model is divided as two subsystems: a surge subsystem including equations (1) and (2) and a disturbance subsystem (equation (3)), it is clear that those two subsystems are linearly independent. Also, from equation (3), one can show that, as a sufficient condition, the nonaxisymmetric disturbance can be suppressed when the extent of the mean velocity trajectories can be restricted within a small neighborhood of the operating point provided the disturbance subsystem is linearly stable. Henceforward, a question is raised: can we control the extent of the trajectory of the surge subsystem, which can be treated as a pure surge system with "unknown" persistent disturbance term, so that the mean velocity would remain in a small neighborhood of the operating point as we desire? This inspires the author to look for a new nonlinear controller which can directly shrink the size of the trajectories in which a pure surge system surges.

This paper is outlined as follows: In section 2, a review of models of an axial compression system is presented. In section 3, the concept of practical stability is introduced followed by a discussion of its application in the one-mode truncated MG system. Starting from section 4, the author presents the detail analysis of the practical stability of the pure surge system using the classical Lyapunov direct method. A new nonlinear control scheme is proposed with numerical validation, which successfully shrinks the size of the trajectories of the pure surge system.

The contribution of this paper are: firstly, a new control strategy, stabilization on the sense of practical stability, is proposed, which may open a door to actively stabilize axial compression systems regardless their instability modes, secondly, a new control scheme is designed to directly shrink the size of the trajectories of the pure surge system, which is the first in its kind in the literature, thirdly, the analysis presented in this paper reveals the importance of the nonlinearity of the compressor performance curve and, in order to control the system in the sense of practical stability, the control scheme must be able to alter the effect of the slope of the compressor characteristic that acts as a nonlinear damping in the system. Because the control scheme designed for the pure surge system in this paper can directly be applied to the surge subsystem (Lin, 1995), the results of this paper are valuable in more realistic cases in which both rotating stall and surge are possible instability modes of the compression systems.

2. MODEL OF A PURE SURGE SYSTEM

The one-mode truncated Moore-Greitzer model equations can be written as following:

\[
\frac{dx_1}{d\tau} = -(x_2 + P_e) + U_e(x_1 + U_e - 1)
\]

\[
- \frac{3}{4}(x_1 + U_e - 1)x_3
\]

\[
(1)
\]

\[
\frac{dx_2}{d\tau} = \frac{1}{\beta_T} [(x_1 + U_e) - U_T(x_2 + P_e)]
\]

\[
(2)
\]

\[
\frac{dx_3}{d\tau} = K_T(1 - (x_1 + U_e - 1)^3)x_3 - x_3^2,
\]

\[
(3)
\]
where $z_1$ and $z_2$ are the disturbances of mean axial velocity and overall total to static pressure rise respectively. $z_3$ represents the square of the amplitude of the first mode of the nonaxisymmetric disturbance, and is always positive. $r$ is the scaled time. The point $(U_r, P_r)$ represents the steady axisymmetric solution of the system, which can be obtained by solving

$$
\begin{align*}
    P_r &= P_r(U_r - 1) \\
    U_r &= U_T(P_r)
\end{align*}
$$

The model equations for a pure surge system are:

$$
\begin{align*}
    \frac{dz_1}{d\tau} &= -(y_s + P_e) + P_r(z_s + U_s) \\
    \frac{dy_s}{d\tau} &= \frac{1}{\beta^2} [(z_s + U_s) - U_T(y_s + P_e)]
\end{align*}
$$

where $U_s = U_r - 1$. $z_s$ and $y_s$ represent the mean velocity disturbance and the overall pressure disturbance respectively.

$P_r(U)$ and $U_T(P)$ are the compressor characteristic and the throttle characteristic respectively. In this research, they are chosen as following:

$$
\begin{align*}
    P_r(z_s + U_s) &= P_{r0} + 1 + \frac{3}{2}(x_s + U_s) - \frac{1}{2}(z_s + U_s)^3 \\
    U_T(y_s + P_e) &= \sqrt{y_s + P_e},
\end{align*}
$$

where $\gamma$ is the throttle coefficient and $\beta$ is a constant related to the Greitzer's B parameter and the steepness of the compressor characteristic. The shut-off head of the compressor, $P_{r0}$, is equal to 1.5.

We use an injection/suction valve placed in front of the compressor as the actuator to implement control rules that we develop based on our nonlinear analysis. Figure 1 is a sketch schematically such a device as well as the rest of the components of the axial flow compression system. Since we only feed back the mean velocity disturbance, the jets through the injection/suction valve are uniform in circumferential direction. The dynamic model equations for such a system are

$$
\begin{align*}
    \frac{dz_s}{d\tau} &= -(y_s + P_e) + P_r(z_s + U_s) \\
    \frac{dy_s}{d\tau} &= \frac{1}{\beta^2} [(z_s + U_s) - U_T(y_s + P_e)]
\end{align*}
$$

where $U_j$ is the velocity of injection/suction whose direction is perpendicular to the axial direction of the compression system. We also specify that negative $U_j$ means additional mass flow is added into the compression system, which is called injection. $A_j$ is the area of the injection/suction valve and $A$ is the area of the compressor duct. $l_i$ is the length of the inlet and $l_e$ is the total length of the inlet, the compressor and the exit duct. The detailed derivation of all of these equations can be found in Lin(1995).

3. CONCEPT OF PRACTICAL STABILITY

Figure 1: The sketch of schematic compression system with an injection/suction valve in front of compressor.

For review purposes, we outline the definitions and the related theorem of practical stability in this section, which are originally presented in [Lasalle & Lefschetz, 1961]. Additional information is contained in [Lakshmikantham, 1990].

Consider the system

$(F^*) \quad \dot{x} = X(x, t), \quad t \geq 0$

with the equilibrium state: $X(0, t) = 0$ for all $t \geq 0$.

The system perturbed by an additional term $p(x, t)$ is

$(F) \quad \dot{x} = X(x, t) + p(x, t), \quad t \geq 0$

We define system $(F^*)$ is

$(PS1)$ Practically Stable if, given a number $\delta$ and two closed and bounded sets $Q_0$ and $Q$ with $Q_0 \subseteq Q$, for each $\delta > 0$, there exists a $T > 0$ and each $t_o \geq 0$, the solution $(F) x^* (t, x_0, t_o)$ is in $Q, t \geq t_o$.

$(PS2)$ Strongly Practically Stable if (i) it is practically stable and (ii), for each $x_0$, each $p$ satisfying $\|p(x, t)\| < \delta$ and each $t_o \geq 0$, the solution $(F^*) x^* (t, x_0, t_o)$ is in $Q$ for $t \geq t_o + T$.

$(PS3)$ Practically Unstable if $(PS1)$ fails to hold.

The concept of practical stability is relative to the number of $\delta$ and the set $Q$ and $Q_0$. The set $Q$ is a set of all the desired states of the system. The subset $Q_0$ is a set of initial states. Before one can speak of practical stability, one must decide: (1) what is the state we desire in practice — the set $Q$; (2) the magnitude of the perturbation to be expected — the value of $\delta$; (3) the range of initial conditions that can be estimated — the set $Q_0$. The conditions which insure the practical stability are neither weaker nor stronger than those insuring regular stability. On one hand, an equilibrium can be regularly stable but unstable in a practical sense. On the other hand, it can be practically stable without being regular. Practical stability is a uniform boundedness of solutions relative to the set of initial conditions $Q_0$ and the class of perturbations $P$. It requires, however, not only that a bound exists but also that the bound be sufficiently small: the solutions starting within $Q_0$ are to remain in $Q$.

It is clear that an axial compression system can conceptually be modeled as $(F^*)$, where $p(x, t)$ represents all possible persistent disturbances such as inlet distortion. In order to apply the concept of practical stability to axial compression systems, we have to estimate the initial set $Q_0$, the upper limit $\delta$ and...
the set of desired states $Q$. Specially, when the system model is simplified as a 3-state model (the one-mode truncated MG model) without inlet distortion, the set of $Q_0$ can be stated as $\{Q_0 : x_1(0) = x_2(0) = 0.0, x_3(0) \leq C\}$, where $C$ is a finite, positive number, because experiments show that the compressor always lose its stability to the nonaxisymmetric disturbance first even though the system eventually surges. The number of $\delta$ is zero in this case. The set of $Q$ is also strongly dependent on experimental situations, and for lack of information it would not be specified in this paper. The advantage of introducing the concept of practical stability is that, in the sense of practical stability, the practically unstable mode of the compression systems becomes unique, that is, for whatever reasons, if a trajectory starting inside $Q_0$ goes outside $Q$, the system is practically unstable. Therefore, if successful, a controller designed based on the concept of practical stability would be able to stabilize the system regardless the instability modes of the system.

In determining practical stability, linear approximations are clearly not appropriate. However, there is no systematic method available to analytically determine practical stability due to the complexity of nonlinear systems. One tool which we will use is the theorem of LaSalle and Lefschetz, which gives a sufficient condition for strong practical stability by means of Lyapunov functions.

Theorem of LaSalle and Lefschetz: Let $V(z)$ be a scalar function which for all $x$ has continuous first partial derivatives and with the property that $V(x) \to \infty$ as $\|x\| \to \infty$. Let $V_1 = \max_{x \in Q_0} V(x)$ for all $p$ satisfying $\|p(x,t)\| < \delta$, and all $t \geq 0$ and if $V(x) \leq V(y)$ for all $x$ in $Q_0$ and all $y$ outside $Q$, then $(F^\infty)$ possesses a strong practical stability.

For the reasons stated in the introduction, if the surge subsystem, equations (1) and (2) which can be treated as a pure surge system undergoing a persistent disturbance due to the nonlinear interaction between the mean flow and the nonaxisymmetric disturbance, can be controlled so that it behaves as we desire, the whole 3-state system will be stabilized in the sense of practical stability. In this paper, the focus is on the pure surge system. It is demonstrated by [Lin, 1995] that the application of the results of this paper to the 3-state system is straightforward.

In following two sections, the theory of practical stability is applied to analyze the dynamics of the pure surge system. This analysis does not aim at quantitatively evaluating whether the system is practically stable under various parameter values. Instead, it is performed to qualitatively find out the factors which determine the practical stability of the pure surge system, so that a nonlinear controller can be designed based on the results of the analysis.

4. ANALOGY BETWEEN THE PURE SURGE SYSTEM AND VdP OSCILLATOR

The model equations of the pure surge system, equation (4) and (5), can be rewritten as a single second order equation,

$$\dot{x} + f(x)\dot{x} + g(x) = h(x, \dot{x}, t).$$

In fact, the well-known Van del Pol equation

$$\dot{x} + \mu x^2 \dot{x} + \beta x = p(x, \dot{x}, t)$$

is another particular form of Liénard's equation (9).

Physically speaking, this means that the surge phenomenon in the axial compression system is analogous to the VdP oscillator with different nonlinear damping. This fact was first observed by Greitzer [1976]. The VdP equation has been studied extensively in recent decades, and the analytical tools for the VdP equation can be applied to the pure surge system.

5. LYAPUNOV FUNCTION OF THE PURE SURGE SYSTEM

Rewrite equation (8) in the following form:

$$\dot{x}_p = y_p$$ (11)

$$\dot{y}_p = -f(x_p)y_p - g(x_p) + h(x_p, y_p)$$ (12)

where

$$x_p = x, \quad y_p = - (y_s + P_s) + P_c (x_s + U_s - 1)$$ (13)

$$f(x_p) = - \frac{dP_c}{dx_p} = - \frac{3}{2} \left[1 - (x_p + U_c)^2\right]$$

$$g(x_p) = \frac{1}{\beta^2} x_p$$

$$h(x_p, y_p) = h(y_s) = - \frac{1}{\beta^2} \left[U_s - \sqrt{y_s + P_s}\right]$$

The term $h(x_p, y_p)$ serves as an "unknown" perturbation "force" in our pure surge system like the one in system ($F^\infty$). The boundedness of $h$ here is guaranteed physically, not mathematically. First of all, because the physical meaning of $y_s$ is the disturbance of total to static pressure rise across the compression system, it must be larger than or equal to $-P_s$. Otherwise, the overall pressure rise $P(t)$ would be less than zero at the instant of $y_s < -P_s$, which means the compressor at that moment would act as a turbine. This is impossible in practice. On the other hand, $y_s$ cannot be infinite because otherwise the system would blow up. Therefore according to the definition of $h$, it must be finite for any physically meaningful value of $y_s$. If we denote $\delta$ as the maximum value of $|h|$ for physically meaningful values of $y_s$, we then conclude from the above argument that

$$|h(y_s)| \leq \delta.$$ 

Certainly, this is also true for all of the physically meaningful values of $x_p$ and $y_p$ because $(x_p, y_p)$ is simply a coordinate transformed from $(x_s, y_s)$.

From our experience of analyzing the VdP equation, we select the following scalar function for the practical stability analysis of the pure surge system:

$$V_p = \frac{1}{2} y_p + F(x_p) - a(x_p)^2 + \frac{1}{2} y_p^2 + 2G(x_p)$$
where

\[ F(x_p) = \int_0^{x_p} f(u)du = -3 \frac{1}{2} \left( x_p - \frac{1}{3}(x_p + \bar{U}_e)^3 + \frac{1}{3} \bar{U}_e^3 \right) \]

\[ G(x_p) = \int_0^{x_p} g(u)du = \frac{1}{\beta^2} x_p^2 \]

\[ \alpha(x_p) = \begin{cases} 3x_p & a_- \leq x_p \leq a_+ \\ 3a_+ & x_p > a_+ \\ 3a_- & x_p < a_- \end{cases} \]

and \( a_-, a_+ \) are constants to be determined. It is clear that \( V_p \) has continuous first partial derivatives with the property that \( V_p(x_p, y_p) \to \infty \) as \( ||(x_p, y_p)|| \to \infty \).

The time derivative of this scalar function can be computed and after substitution of equations (11) and (12) we get

\[ \dot{V}_p = -[f(x_p) + \alpha'(x_p)]y_p^2 + [2h(x_p, y_p) - \alpha'(x_p) \{F(x_p) - \alpha(x_p)\}]y_p - [g(x_p) - h(x_p, y_p)] \{F(x_p) - \alpha(x_p)\} \]

where \( \alpha'(x_p) \) denotes \( \frac{d\alpha}{dx_p} \).

We do not intend to determine whether the pure surge system is practically stable because it is impossible to do so before we have clear and exact knowledge about \( \delta, Q_0 \) and \( Q \). Instead, we use the theory of practical stability here to find out which factors affect the extent of the solution trajectories of the system. As long as this can be done, we then can design controller to control the extent of the solution trajectories such that the practical stability of the pure surge system can be as strong as we prefer.

In other words, we need to find out which factors determine the size of the two sets \( Q_0 \) and \( Q \) satisfying the theorem of LaSalle and Lefschetz. Let's agree to select \( Q_0 \) as a rectangular region: \( a_- \leq x_p \leq a_+ \). Then our aim is to select or determine the values of \( a_-, a_+, b_-, b_+ \) such that \( V_p \leq -\varepsilon < 0 \) for all \((x_p, y_p)\) outside \( Q_0 \). After \( Q_0 \) is found, \( Q \) will be the region defined by \( V_p \leq V_0 \), where \( V_0 \) is the maximum value of \( V_p \) when \((x_p, y_p)\) in \( Q_0 \).

Let's begin with the observation of the function

\[ \mathcal{F}(x_p) = F(x_p) - 3x_p \]

\[ = \frac{1}{2}(x_p + \bar{U}_e)^3 - \frac{9}{2} x_p - \frac{1}{2} \bar{U}_e^3. \]

In the case of \(-\sqrt{3} < \bar{U}_e < \sqrt{3}\), the curve of \( \mathcal{F}(x_p) \) can be plotted qualitatively as in figure 2, which indicates that the equation \( \mathcal{F}(x_p) = 0 \) has three real roots, denoted as \( x_- \), \( x_0 \), and \( x_+ \), with \( x_- < 0 < x_+ \). Remember that we are only interested in the situation in which the operating point is close to the peak of the compressor performance curve, so the operating region characterized by \(-\sqrt{3} < \bar{U}_e < \sqrt{3}\) will cover all the region of interest for the particular compressor characteristic we choose for this research. Let \( a \) be a real number. We note that \( F(a) > 0 \) for \( a > x_+ > 0 \) and \( \mathcal{F}(a) < 0 \) for \( a < x_- < 0 \). Moreover, by selecting \( a > x_+ \) (\( a < x_- \)) but as close to \( x_+ (x_-) \) as possible, we have \( F(a) > 0 \) (\( F(a) < 0 \)) but as small (large) as we please. For \( x_- < a < x_+ \), the maximum and the minimum values of \( \mathcal{F}(a) \) are denoted as \( \mathcal{F}_{max} \) and \( \mathcal{F}_{min} \) respectively, and henceforward the maximum of \( |\mathcal{F}(a)| \) for \( x_- < a < x_+ \) is the larger value of \( \mathcal{F}_{max} \) and \( |\mathcal{F}_{min}| \), denoted as \( \mathcal{F}_M = \max \{\mathcal{F}_{max}, |\mathcal{F}_{min}|\} \). We select \( a_- > x_+ \) such that \( \mathcal{F}(a_-) = \mathcal{F}_{max} \), and \( a_- < x_- \) such that \( \mathcal{F}(a_-) = \mathcal{F}_{min} \) (figure 2). It is straightforward to determine that \( a_- = 2\sqrt{3} - \bar{U}_e \) and \( a_+ = -2\sqrt{3} - \bar{U}_e \). Since the curve of \( \mathcal{F}(x_p) \) can be divided into three segments: \( x_p < a_- \), \( a_- < x_p < a_+ \), and \( x_p > a_+ \) we then evaluate the sign of \( \dot{V}_p \) in these three cases. After some algebra we have the following results:

**CASE 1** \( x_p > a_+ \):

\[ \dot{V}_p \leq -\frac{2}{33} \left[ \mathcal{F}_{max}^2 - \alpha^2 \right] + \frac{33}{8} \mathcal{F}_{max}^2 - \frac{a_-^2}{\beta^2} \]

**CASE 2** \( x_p < a_- \):

\[ \dot{V}_p \leq -\frac{2}{33} \left[ \mathcal{F}_{min}^2 - \alpha^2 \right] + \frac{33}{8} \mathcal{F}_{min}^2 - \frac{a_-^2}{\beta^2} \]

**CASE 3** \( a_- < x_p < a_+ \):

\[ \dot{V}_p \leq -\frac{3}{2} y_p^2 + (2d + 3F_M)|y_p| + \delta F_M - \min[g(x_p, \mathcal{F}(x_p))] \]

where \( \min[g(x_p, \mathcal{F}(x_p))] \) represents the minimum value of \( g(x_p) \mathcal{F}(x_p) \), which occurs inside the interval of \( a_- < x_p < a_+ \) as long as \(-\sqrt{3} < \bar{U}_e < \sqrt{3}\), and also is less than or equal to zero. i.e. \( \min[g(x_p, \mathcal{F}(x_p))] \leq 0 \).

Define

\[ c_1 = \frac{1}{3} \left( 2d + 3F_M \right) \]

\[ c_0 = \frac{2}{3} \left( \delta F_M - \min[g(x_p, \mathcal{F}(x_p))] \right) \]

so that

\[ \dot{V}_p \leq -\frac{3}{2} \left[ y_p^2 - 2c_1 |y_p| - c_0 \right] \]

Therefore,

\[ \dot{V}_p \leq -\varepsilon < 0 \]
for \( a_- \leq x_p \leq a_+ \) and \( |y_p| \geq c_1 + \sqrt{c_1^2 + c_0} \). In other words,
\[
b_+ = c_1 + \sqrt{c_1^2 + c_0}
\]
and
\[
b_- = -b_+.
\]
Note that \( c_1, c_0 \) are always positive, so \( b_+, b_- \) are always real.

For the cases of \( x_p > a_+ \) and \( x_p < a_- \), we know from (14) and (15) that \( V_{fp} \leq -\epsilon < 0 \) and \( V_{fn} \leq -\epsilon < 0 \) for all \( \delta \) if \( \beta^2 \geq \frac{3 \gamma^2}{8 \epsilon_m} \) where \( \epsilon_m = \max \{a_- - a_+\} \). If \( \beta^2 \leq \frac{3 \gamma^2}{8 \epsilon_m} \), then the inequalities \( V_{fp} \leq -\epsilon < 0 \) and \( V_{fn} \leq -\epsilon < 0 \) are true only for some values of \( \delta \). Hence, for a given \( \delta \), we can always select \( \beta \) and \( \gamma \), the throttle coefficient, such that \( V_p \leq -\epsilon < 0 \) outside the region
\[
\{Q_0 : a_- \leq x_p \leq a_+, b_- \leq y_p \leq b_+\}.
\]
Let's denote
\[
b_m = \max \{b_-, b_+\}.
\]
Then the maximum value of \( V_p \) when \( (x_p, y_p) \) in \( Q_0 \) is
\[
V_0 = \frac{1}{2} [b_m + F(a_m) - 3a_m] + \frac{1}{2} b_m^2 + 2G(a_m).
\]
Then \( Q_0 \) is contained inside the set \( Q \) defined by \( V_p < V_0 \).

At this point we have found sets \( Q_0 \) and \( Q \) which satisfy the theorem of practical stability. The most interesting feature of this analysis is that the interval between the two boundaries of \( Q_0 \) in \( x_p \) direction, that is the interval between \( a_+ \) and \( a_- \), is determined solely by the shape of \( F(x_p) \), which is in turn determined by the nonlinearity of the compressor characteristic. Therefore, we conclude that the nonlinearity of the compressor characteristic is a key factor to determine the practical stability. Physically speaking, this means that the compressor, which acts as a nonlinear damper for the entire compression system, not only affects the dynamics of the system in the neighborhood close to the equilibrium operating points, but also determines the global behavior of the system. This is also the fundamental difference between this practical stability analysis and the linear analyses or some nonlinear analyses based on local dynamics of the equilibrium points.

This conclusion suggests a way to control the surge globally. The interval between \( a_+ \) and \( a_- \) can be altered only by changing the shape of \( F(x_p) \), which must be implemented by an additional control actuator. If the value of \( a_+ \) and \( a_- \) is controlled to be small enough that \( \{Q : V_p \leq V_0\} \) is an acceptable neighborhood of the equilibrium point, then we know from the practical stability theorem that the system possesses a strong practical stability. This means any initial disturbance, no matter whether it is inside \( Q_0 \) or not, will finally enter into \( Q \). The system, henceforward, is robust to any disturbances.

Certainly, the parameter \( \beta \), which is proportional to Greitzer's parameter \( B \), is also important in such a control process. No matter how much \( a_+ \) and \( a_- \) are reduced, the condition that \( V_p \leq -\epsilon < 0 \) outside \( Q_0 \) must carefully be maintained. The smaller \( \beta \), the harder this is to achieve. This can be explained by observing the conditions under which \( V_p \leq -\epsilon < 0 \) outside the region \( Q_0 \) holds. When \( a_m \) is reduced, \( \frac{3 \gamma^2}{8 \epsilon_m} \) increases. So the smaller \( \beta \), the easier it would be for \( \beta^2 \leq \frac{3 \gamma^2}{8 \epsilon_m} \) to occur. If \( \beta^2 \leq \frac{3 \gamma^2}{8 \epsilon_m} \) did hold, some constraints on \( \delta \) should be added in order to keep \( V_p \leq -\epsilon < 0 \) outside the region \( Q_0 \). Since \( \delta \) is uncontrollable in this analysis, the surge with smaller \( \beta \) then is more difficult to be suppressed. However, this does not contradict with the fact that larger \( \beta \) implies that the uncontrolled system is easier to surge. Surge with larger \( \beta \) is easier to occur in the uncontrolled systems, and easier to be suppressed in the controlled system.

Unfortunately, due to lack of information about \( \delta \), the analytical procedures have to stop here. In the next sections, a nonlinear controller corresponding to different kinds of actuators will be designed based on the above conclusion. Numerical simulations will verify that the results of the above analysis are valid for the model system (4) and (5) regardless the choices of the Lyapunov function.

6. NONLINEAR CONTROL OF SURGE

In order to design a nonlinear controller so that the practical stability of the controlled system can be as strong as we prefer, we again go through the algebra to rewrite equations (6) and (7) as a single second order equation. The result is
\[
\ddot{z}_s - \frac{dP_s}{dz_s} \dot{z}_s + \frac{d}{dr} \left[ \left( \frac{l_1}{l_e} \right) \frac{du_j}{dr} + (z_s + U_s)u_j \right] + \frac{1}{\beta_0^2} \ddot{z}_s = h(y_s)
\]
where
\[
u_j = \left( \frac{A_j}{\Lambda} \right) U_j.
\]
Our goal here is to find an appropriate feedback law so that the damping term can be controlled. One way to do it is to make the third term of the above equation have a form of \( f_1(z_s) \dot{z}_s \) by selecting an appropriate feedback law. In other words, let's assume that
\[
\frac{d}{dr} \left[ \left( \frac{l_1}{l_e} \right) \frac{du_j}{dr} + (z_s + U_s)u_j \right] = f_1(z_s) \dot{z}_s
\]
where \( f_1(z_s) \) is to be determined. We now have
\[
\ddot{z}_s - \frac{dP_s}{dz_s} f_1(z_s) \dot{z}_s + \frac{1}{\beta_0^2} \ddot{z}_s = h(y_s).
\]
After going through the same analytical procedures as those presented in last section, we find that one of the ways to alter the nonlinear damping term so that the practical stability can be enhanced is to choose
\[
f_1(z_s) = \frac{3}{2} \left[ (Z^2 - 1)z_s^2 + 2(U_s - 1)(Z - 1)z_s \right],
\]
where \( Z \) is the control constant which can be adjusted by the designer for obtaining preferred performances. \( Z = 1 \) means no control. In fact, the above result indicates that the feedback law, \( u_j(z_s) \), must be obtained by simultaneously solving the original governing equations (6), (7) and the following equation:
\[
\frac{d}{dr} \left[ \left( \frac{l_1}{l_e} \right) \frac{du_j}{dr} + (z_s + U_s)u_j \right] = \frac{3}{2} \left[ (Z^2 - 1)z_s^2 + 2(U_s - 1)(Z - 1)z_s \right] \frac{dz_s}{dr},
\]
\[17\]
Figure 3: The three design operating points at which the uncontrolled system is practically unstable.

One of the initial conditions for \( u_j \) is

\[ u_j(0) = 0 \]

because there must be no control at \( t = 0 \). The other initial condition for \( u_j \) is obtained by integrating (17) once and substituting \( z_k(0) \) into it. The result is the following:

\[
\left( \frac{du_j}{dt} \right) \bigg|_{t=0} = \frac{1}{2} (Z^2 - 1)z^2_k(0) + \frac{3}{2} (U_1 - 1)(Z - 1)z^3_k(0). \tag{18}
\]

Thus, the model equations for the pure surge system with additional injection/suction valve in front of the compressor are well posed.

7. NUMERICAL EXAMPLES

To illustrate the effectiveness of our nonlinear controller, we select three typical practical unstable cases which correspond to the three design operating points in figure 3. Point I is the case of surge-like instability, that is, a small disturbance on mean velocity or/and overall pressure rise would make the system surge over one cycle and then return to the design operating point and stays there. Point II and III are the cases in which the design operating point of the pure surge system is unstable and the system itself has a stable limit cycle. In figure 3, the dotted portion of the compressor characteristic means that the design operating point located in this portion are asymptotically unstable. We are going to show that our controllers can effectively shrink the size of the limit cycle in the last two examples, and then stabilize the system in a practical sense. In the first example, our controllers would shut off if \( Z = 1.0 \), because from equations (17) the controller is shut off if \( Z = 1.0 \). In figure 4, one can see that surge-like instability occurs in the uncontrolled system but disappears when the controller is turned on. The extent of the solution trajectories become smaller and smaller when increasing the control coefficient \( Z \). If the set of desired states \( Q \) is given, we could always make the solution trajectories finally stay inside \( Q \) by choosing an appropriate value of \( Z \), because the controlled system possesses a strong practical stability. In that case, the system is not only asymptotically stable but also strongly practically stable.

In figure 5, the time history of each controller action is presented. The ratio \( U_j / U \) represents the ratio of the additional mass flow rate and the design mass flow rate. The larger the number, the more control energy we need. We see from this figure that the maximum additional mass flow rate we need to supply can be less than 2.5% of the design mass flow rate (Z = 40 case). So the control energy of the controller is very small. Since the rate of change \( u_j \) is also important to implementation of the control scheme, the time history of \( \frac{du_j}{dt} \) with respect of different values of \( Z \) is provided. It is observed that the values of \( \frac{du_j}{dt} \) are also reasonably small.

Example 2: Control of Surge (point II) The parameters \( \beta \) and \( \gamma \) in this example are 4.0 and 1.2 respectively. As shown in figure 3, the design operating point \( (U_1, P_1) \) is very close to the peak of the compressor characteristic but is asymptotically
We present the numerical results in figure 6 and figure 7. Again \( Z = 1.0 \) means no control. The controllers can not change the regular stability of the design operating point because the control laws are completely nonlinear and therefore the linear behavior remains with no change. However, the size of the limit cycles are shrunk down by the controllers. If the set of desired states \( Q \) is given and the stable limit cycle is controlled to be small enough to fit into the set \( Q \), then the system is practically stable even though it is regularly unstable.

Example 3: Control of Surge (point III) The parameters \( \beta \) and \( \gamma \) in this example are 4.0 and 0.8 respectively. As shown in figure 3, the point III now is in the middle of the deep surge region and the design operating point is far away from the peak of the compressor characteristic. We present the numerical results in figure 8 and figure 9. The results show that even in this case, the controllers still stabilize the system in the sense of practical stability. However, we note that in order to control the system the maximum additional mass flow rate we need to supply is now about 5.0\% of the design mass flow rate (\( Z = 40 \) case).

8. CONCLUSIONS

In this paper, a new control strategy, stabilization on the sense of practical stability, is proposed. The advantage of introducing the concept of practical stability is that, in the sense of practical stability, the practically unstable mode of axial compression systems become unique. A controller designed based on this control strategy would be able to stabilize the system regardless its instability modes. As the first step, the theory of practical stability is applied to analyze the pure surge system, a compression system without any nonaxisymmetric disturbance. It is found that the nonlinearity of the compressor characteristic, which is the damping term in this second order, nonlinear oscillatory system, plays a key role in determining the practical stability of the system. Based on this result, a nonlinear controller is designed to modify the nonlinear damping in the pure surge system by feeding back the mean velocity disturbance to the actuator, an injection/suction valve placed in front of the compressor. Numerical simulations show that this controller successfully stabilize the system in the sense of practical stability.

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Figure 9: Numerical solution of the controlled pure surge system with the injection/suction valve — Example 3: time history of the actuator action.

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