AN ANALYTICAL MODEL OF AXIAL COMPRESSOR OFF-DESIGN PERFORMANCE

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ABSTRACT

An analysis is presented of the off-design performance of multistage axial-flow compressors. It is based on an analytical solution, valid for small perturbations in operating conditions from the design point, and provides an insight into the effects of choices made during the compressor design process on performance and off-design stage matching.

It is shown that the mean design value of stage loading coefficient \( \psi = \Delta \theta _0 / U^2 \) has a dominant effect on off-design performance, whereas the stage-wise distribution of stage loading coefficient and the design value of flow coefficient have little influence. The powerful effects of variable stator vanes on stage-matching are also demonstrated and these results are shown to agree well with previous work.

The slope of the working line of a gas turbine engine, overlaid on overall compressor characteristics, is shown to have a strong effect on the off-design stage-matching through the compressor. The model is also used to analyze design changes to the compressor geometry and to show how errors in estimates of annulus blockage, decided during the design process, have less effect on compressor performance than has previously been thought.

INTRODUCTION

The off-design performance of axial-flow compressors is critical for the efficient and stable operation of aircraft gas-turbine engines and industrial compressors. While modern computational methods can give predictions of compressor off-design performance, there remains some lack of understanding of how various features of the original design (e.g., the magnitude of the stage loading coefficient and its distribution through the compressor) affect the off-design performance. The purpose of this paper is to advance that understanding.

A one-dimensional approach is developed here, similar in this respect to many that have proved useful in the past (e.g., Robbins and Dugan (1956), Stone (1956), Howell and Calvert (1978), Wright and Miller (1991)). Critically, however, we make an assumption that allows the stage-stacking equations to be solved analytically. Our assumption is that changes in gas properties occur continuously with axial distance through the compressor and not in discrete jumps between blade rows, an assumption which is similar to an original idea of Mellor (1957), Horlock (1958) developed Mellor's concept to obtain and solve a differential equation for the temperature change through a turbomachine when it operated off-design. More recently, Goede and Casey (1988) have adopted a somewhat similar approach to the Mellor/Horlock solutions. They solved the basic stage-stacking equations, assuming that each stage deviates only slightly from its design operating point. The resulting equations are solved numerically and the authors trace out how each stage moves away from its design point when an off-design stage loading distribution is specified. Particularly enlightening in the Goede/Casey analysis is the effect of varying the stagger of stators through the compressor, and how this feature can be used to keep compressor pressure ratio and efficiency near design values.

Here we bring together the Mellor/Horlock analysis (obtaining a differential equation for the temperature rise through the equivalent compressor) and the Goede/Casey method of solution (assuming small perturbations of all the independent and dependent parameters). We obtain analytical solutions for the temperature, density and axial-velocity variations through the machine, which enable us to deduce how the choice of design parameters affects the off-design performance of individual stages and of the overall compressor.

Inherent in the method are two further assumptions, namely that the temperature rise coefficient - flow coefficient characteristic for each stage and

NOMENCLATURE

\( A = \) annulus area \( \alpha = \) absolute flow angle
\( \psi = \) specific heat at const. press. \( \beta = \) relative flow angle
\( K = \) slope of stage characteristic \( \gamma = \) ratio of specific heats
\( M = \) mass flow rate \( \eta = \) efficiency
\( N = \) number of stages \( \theta = \) non-dim. exit temp.
\( T = \) static temperature \( \rho = \) density
\( U = \) mean blade speed \( \phi = \) flow coefficient (\( V_x U \))
\( V_x = \) axial velocity \( \psi = \) loading coeff. (\( \Delta \theta _0 U^2 \))
\( h = \) enthalpy \( \omega = \) non-dim. temperature
\( k = \) small perturbation
\( \tau = \) temperature ratio
\( x = \) axial distance

Constants: \( a, b, n, P, Q, R, S, W, \delta, \xi, \iota, \kappa, \lambda, \mu, v, \sigma, \chi \)

Subscripts: \( I \) compressor inlet
\( II \) compressor exit
\( o \) stagnation condition

Superscripts: \( * \) design condition
\( \cdot \) off-design condition
\( \prime \) first derivative
\( \cdot \cdot \) second derivative

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stage is linear and that the stage efficiencies are constant between design and off-design conditions. These assumptions limit our solutions to operating points close to the compressor design point. However, comparisons of our solutions with a numerical method (Hynes, 1992), which takes a fuller account of the performance changes of blade rows with incidence, shows that our approximations are good. Comparisons with the results of Goede and Casey show similar predictions of compressor behaviour with variable stator movement.

It is considered that the analytical solutions obtained give an improved understanding of the nature of off-design performance. Among several applications of the solutions, equations are developed which predict the slope and spacing of constant speed characteristics and which show the effects of errors in annulus blockage estimates and the slope of the engine working line on the matching of individual stages.

METHOD

In this section the basis of the analysis is outlined. After a description of the necessary assumptions, the method of solution is presented using a limited set of variables. Finally the solution for a fuller set of variables is given.

It is first necessary to make five assumptions, which are described below.

1) Only small changes from the compressor design point are considered (the change in any stage flow coefficient is limited to a maximum of four percent from its design value).

2) The stage loading - flow coefficient characteristics for each stage of the compressor are assumed to be linear. For the uninstalled operating range, the stator exit flow angle, \( \alpha_1 \), and the rotor exit relative flow angle, \( \beta_2 \) (both measured from the axial direction) are assumed to be constant with changing flow coefficient. Hence

\[
\psi = \frac{C_p n}{U^2} = 1 - \frac{V}{U} \left( \tan \alpha_1 + \tan \beta_2 \right) = 1 - \phi K
\]

where \( \psi \) is the stage loading coefficient, \( \phi \) is the flow coefficient and \( K = \tan \alpha_1 + \tan \beta_2 \), the negative slope of the characteristic which is assumed to be constant. In practice, the outlet flow angles will increase slightly with incidence and the \( \psi - \phi \) characteristic will "droop" a little as the stall point is approached.

3) The efficiencies of each stage are assumed to be equal and, for the small off-design perturbations considered, they are assumed not to change between the design and off-design conditions. This implies that a constant polytropic efficiency (\( \eta_p \)) can be used. In practice, for small changes in \( \phi \) from design, the stage efficiency changes relatively little (less than 1% for a 10% change in flow coefficient from design in an example given by Howell (1945)).

4) To obtain analytical solutions to the off-design equations given below it is necessary to assume that the design value of stage loading coefficient (\( \psi^* \)) is the same for every stage in the compressor. A study of several modern compressors showed this assumption to be close to common design choices. In a later section we compare the effects of other distributions of design stage loading through the compressor by solving the governing equations numerically.

5) The changes in gas properties are assumed to occur continuously with axial distance through the compressor, as discussed in the introduction.

Off-Design Performance of a "Baseline" Compressor

Consider first the performance of a "baseline" compressor using the following variables: \( \psi, \phi, p \) (density), \( T \) (temperature), \( U \) (speed) and \( M \) (mass flow rate). We denote design values by the superscript "\( \ast \)." For simplicity in this first case we take \( U \) and \( M \) to be the same at every axial position through the compressor. Consider small perturbations, \( k(x) \), in the design value of each parameter, where \( x \) is the axial distance through the compressor. (We choose the scale of \( x \) such that one unit of axial distance is occupied by one stage. Thus an axial compressor of \( N \) stages has a length \( x = N \).)

\[
\frac{\psi(x)}{\psi^*(x)} = 1 + k\psi(x) \quad \frac{\phi(x)}{\phi^*(x)} = 1 + k\phi(x)
\]

\[
\frac{\rho(x)}{\rho^*(x)} = 1 + kp(x) \quad \frac{T(x)}{T^*(x)} = 1 + kT(x)
\]

\[
\frac{U}{U^*} = 1 + kU \quad \frac{M}{M^*} = 1 + kM
\]

where each of the perturbation terms \( k \) has a magnitude which is much less than unity. Note that for this baseline case, \( kU \) and \( kM \) are constants and are independent of \( x \). These parameters are related by four equations, as follows.

Since the polytropic efficiency of each stage is assumed to remain constant

\[
\frac{\rho(x)}{\rho^*(x)} = \left[ \frac{T(x)}{T^*(x)} \right]^0
\]

where \( n = \eta_p \frac{V}{U^2} - 1 \). For small perturbations this becomes

\[
k\phi(x) = n kT(x)
\]

Consideration of mass continuity at design and off-design gives

\[
\frac{M}{M^*} = \frac{\phi}{\phi^*} = \frac{U}{U^*}
\]

and for small changes from the design point it follows that

\[
kM = k\rho(x) + k\phi(x) + kT(x)
\]

Equation 1, the relation between \( \psi \) and \( \phi \), may be written, for small perturbations,

\[
k\psi(x) = \left( 1 - \frac{1}{\psi^*} \right) k\phi(x)
\]

Finally, for small movements from the design point and for \( \psi^* \) constant through the compressor, we assume that the ratio of the finite change of temperature across the stage to the finite design change may be represented by small (differential) quantities, so that

\[
\frac{dT}{dx} = \frac{dT^*}{dx} \frac{\Delta T}{\Delta T^*} \frac{\Delta T^*}{\Delta T}
\]

If \( \psi^* \) is constant through the compressor then

\[
T^*(x) = T_1 + \frac{\psi^* U^* + 2x}{C_p}
\]

and

\[
\frac{dT^*}{dx} = \frac{\psi^* U^* + 2x}{C_p}
\]

Now

\[
\frac{\Delta T}{\Delta T^*} = \frac{\psi}{\psi^*} \left( \frac{U^*}{U} \right)^2
\]

which becomes

\[
\frac{\Delta T}{\Delta T^*} = 1 + k\psi(x) + 2kU
\]

Thus a differential equation for the temperature variation from design (\( kT \)) may be obtained from equations 8, 9, 10 and 11.
A More General Solution

We next derive a more general solution in which the compressor design is allowed to change slightly from the baseline design. We follow a procedure similar to that described above, but now include the effects of other input variables. These include

1) varying the slopes of the stage characteristics (possibly as a result of movement of variable stator vanes),

\[
\frac{K(x)}{K^*(x)} = 1 + k_K(x) \quad (17)
\]

2) allowing annulus areas to change from design to off-design (as a function of x), either as a design modification from the baseline compressor or due to boundary layer growth,

\[
\frac{A(x)}{A^*(x)} = 1 + k_A(x) \quad (18)
\]

3) taking account of changes in blade speed due to changes in both rotational speed off-design and changes of mean radius with x, so that \( U = U(x) \),

\[
\begin{align*}
\frac{U}{U^*(x)} = 1 + k_U(x) \\
\frac{M}{M^*(x)} = 1 + k_M(x)
\end{align*} \quad (19)
\]

4) allowing small changes in mass flow due to changes in both entry mass flow and to bleed or injection of air along the compressor, so that \( M = M(x) \),

\[
\frac{M}{M^*(x)} = 1 + k_M(x) \quad (20)
\]

It is possible to obtain an analytical solution if these variables take any polynomial function of \( x \), but the solution quickly becomes very complicated with the increasing order of the polynomials. Here we consider only linear distributions of: \( k_U(x) = \chi_0 + \delta x \), \( k_M(x) = \epsilon_0 + \lambda x \), \( k_K(x) = \mu_0 + \nu x \) and \( k_A(x) = \mu_0 + \nu x \).

For this case equation 13 becomes

\[
k_T(x) = \frac{S}{P} \left[ 1 - \frac{R_{x}}{(R + x)^P} \right] \frac{W(R + x)}{P + 1} \left[ 1 - \frac{R^{P+1}}{(R + x)^{P+1}} \right] \frac{W_x}{P} \quad (12)
\]

where

\[
P = 1 + \frac{\chi_0}{(R + x)}
\]

\[
Q = (1 - \frac{1}{(R + x)}) (k_M - k_U) + 2k_U
\]

\[
R = \frac{C_p T_1}{\gamma^2 + 1}
\]

This linear, first order differential equation is solved using an integrating factor and the boundary condition that \( k_T(0) = 0 \) (the inlet temperature remains constant at \( T = T_1 \)).

Expressions for \( k_T(x) \), \( k_T(x) \) and \( k_T(x) \) follow by back-substitution into equations 4, 6 and 7.

At the exit from the compressor, where \( T = T_{II} \) at \( x = N \),

\[
(R + x) = R \left[ 1 + \frac{N v_* U^*_2}{C_p T_1} \right] = R T_{II}^* \quad (15)
\]

where \( T_{II}^* = \frac{T_{II}}{T_1} \), the ratio of compressor exit to inlet temperatures at design.

Hence

\[
k_T(x) = k_T(N) = \frac{Q}{P} \left[ 1 - \frac{1}{(R^{*})^P} \right] \quad (16)
\]

This solution for the baseline compressor may be shown to be similar to a logarithmic solution obtained by Horlock (1958) if the design temperature ratio \( T_{II}^* \) is not much greater than unity.

VALIDITY OF THE SOLUTION

Two exercises to check the validity of the analytical solution were undertaken. Firstly, the results for changes in speed and flow (equation 14) were compared with a numerical mean-line performance prediction code; and secondly, the effects of changing stator stagger angle (which changes the slope of the stage characteristics) and/or bleed flows.

Comparison of the Solution with Mean-Line Code

To check the accuracy of equation 14, several test cases were compared with the output from a mean-line compressor performance code. This code, which was developed by Dr T.P. Hynes at the Whittle Laboratory, is not limited to small perturbations from the design point, deviation angles and efficiencies being calculated from empirical correlations.

For a hypothetical ten-stage, high speed compressor with a constant speed increase (\( k_{UU} \)) and a small flow increase above the design mass flow rate (\( k_{MM} \)), the speed being held constant, and secondly a small speed increase (\( k_{U} U^* \)) at a constant mass flow rate. Figure 1 compares the changes in flow coefficient (\( k_U \)) calculated with each method for the first test case. Similarly good agreement was found for all parameters for each of the two cases.

To investigate the sensitivity of the solution to the "small perturbation" approximation, the second case was investigated with several different positive values of \( k_{U} \) (0.001, 0.005, 0.01 and 0.05). From these calculations it was decided that, for accuracy, the size of the perturbations of the input variables (\( k_{U} \) and \( k_{M} \)) should be such
that the size of the flow coefficient perturbation ($k_\phi$) should nowhere exceed 4%.

**Comparison of the Solution with the Results of Goede and Casey**

As another exercise to validate our method, comparisons were made with the results of Goede and Casey (1988). They assumed that compressor speed and flow rate were fixed at the design values and they evaluated the distribution of flow coefficient after specifying the off-design stage loading distribution through the compressor. Variable stator vanes allowed the shapes of the stage characteristics to differ one from another. The equations which they derived for single stages were solved numerically to obtain the distribution of flow coefficient through the whole compressor.

The method of Goede and Casey can be thought of as a special case of our analysis. Instead of specifying changes in speed, flow and stage characteristic slope ($k_U$, $k_M$, and $k_K(x)$) as input variables, they specified speed, flow and off-design stage loading ($k_U$, $k_M$, and $k_W(x)$). They observed that the distribution of changes in stage loading from design was approximately linear with stage number for several industrial compressors. Consequently they specified linear distributions of stage loading as an input to their model. Below we show how our new model can provide analytical solutions for the same problem.

After Goede and Casey, assume *a priori* that

$$k_W(x) = a + bx \quad (23)$$

where $a$ and $b$ are constants and are both much less than unity. Substitution into equation 11 yields

$$k_T'(x)(R + x) + k_T(x) = a + bx + 2k_U \quad (24)$$

We solve this equation using an integrating factor and the same boundary condition as before, namely that $k_T(0) = 0$, and obtain

$$k_T(x) = \frac{1}{(R + x)} \left[ (2k_U + a)x + \frac{bx^2}{2} \right] \quad (25)$$

The Goede and Casey analysis assumes that the compressor speed and flow rate are fixed at their design values; therefore we set $k_U = k_M = 0$. Back substitution for $k_W(x)$ gives

$$k_\phi(x) = \frac{-n(k_W^2 - a^2)}{2(Rb - a + k_U)} \quad (27)$$

This relation is plotted in figure 2a for a six stage compressor for the three different distributions of off-design stage loading given in table 1. The origin of this graph corresponds to the compressor design point where, but for any perturbing influence of variable stators, all stage operating points would normally lie. If the stage loading of all stages is increased by 0.1% (case 1) then $k_\phi$ becomes negative as the flow decelerates from its design axial velocity as it passes through the compressor. If we exacerbate the situation by reducing the slope of the stage characteristics with stage number (case 2) then the stage operating points can be seen to move more rapidly from the origin. The third case illustrates a situation where the slopes of the stage characteristics are progressively increased through the machine (negative $b$). The effect of this is to limit the off-design movement of the front stages by counteracting the original increase in stage loading (positive $a$). Figure 2b is reproduced from Goede and Casey (1988). It shows numerical solutions for the same three test cases as above. The agreement between the analytical and numerical solutions is good.
The distribution of change in stage characteristic slope (which could be caused by variable stators) that is required to obtain the specified distribution of \( k_T(x) \) can be found by back substitution to obtain \( k_K(x) \). We find

$$ k_K(x) = \frac{(a + bx)}{(1 - \frac{1}{\psi^*})} + \frac{nx(2a + bx)}{2(R + x)} \quad (28) $$

This relation then gives the stage geometry distribution (the variation in \( K(x) \), possibly due to changes in stator outlet angles \( \alpha_2(x) \)) that would exactly create the linear distributions of \( k_T(x) \) assumed \textit{a priori} by Goede and Casey.

Similar comparisons were made with the Goede and Casey model using results for small changes in compressor speed (Casey, 1992). Again good agreement between the numerical and analytical solutions was obtained.

**THE OFF-DESIGN PERFORMANCE OF THE BASELINE COMPRESSOR**

In this section we examine three aspects of the off-design performance of the "baseline" compressor, described by equations 14 and 16. We begin by considering the importance of the choice of the design conditions (\( \psi^*, \phi^* \)) and then use the equations to derive expressions for the slope and spacing of "overall" characteristics (compressor inlet to exit). Finally we consider the effects of the slope of the compressor working line on stage-matching.

**The Choice of Design Conditions (\( \psi^*, \phi^* \))**

The Effect of the Magnitude of \( \psi^* \). The magnitudes of the stage loading coefficient (\( \psi^* \)) and the overall temperature ratio (\( \eta^* \)) at the design point are the factors which most strongly influence how a compressor operates under off-design conditions. In this section we study how the magnitude of \( \psi^* \) affects how quickly stage operating points move away from their design points. Equation 14 gives the solution for the temperature distribution when we consider changes in rotational speed and flow rate only. Differentiating this equation twice with respect to \( x \) we obtain

$$ k_T''(x) = -\frac{Q}{R^2} \left( P + 1 \right) \left( 1 + \frac{x}{R} \right)^2 \quad (29) $$

For this example let us consider \( K_U > k_M \left( \frac{1-\psi^*}{(1+\psi^*)} \right) \) so that \( Q \) is positive and that \( k_T'(x) \) is positive for all \( x \). (Similar conclusions to those drawn below apply to cases where \( Q \) is negative but some sign reversal is necessary in the arguments.) From equation 29 we see that \( k_T''(x) = 0 \) if \( P = -1 \), and hence if \( \psi^* = n/(n+2) \) the curvature of \( k_T(x) \) is positive whereas for \( \psi^* > n/(n+2) \) the curvature is negative. From equation 14 we see that if \( P > 0 \) is positive then \( k_T(x) \) will tend to \( Q/P \) at large values of \( x \). This is the case if \( \psi^* > n/(n+1) \). We summarise these results in figure 3 which shows \( k_T \) plotted against \( x \) for four different values of \( \psi^* \) (for these examples \( k_U = 0.001 \) and \( k_M = 0 \)). Table 2 provides the key to this figure.

If we assume that \( \eta_p = 0.9 \) and \( \gamma = 1.4 \) then the critical values of \( \psi^* \) are

$$ \frac{n}{n+2} = 0.518 \quad \frac{n}{n+1} = 0.683 $$

<table>
<thead>
<tr>
<th>Case</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+0.001</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>+0.001</td>
<td>+0.005</td>
</tr>
<tr>
<td>3</td>
<td>+0.001</td>
<td>-0.005</td>
</tr>
</tbody>
</table>

Table 1  Three distributions of \( k_T \) as plotted in figure 2a.

Table 2  The four cases of figure 3.

<table>
<thead>
<tr>
<th>Case</th>
<th>( \psi^* )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \psi^* &lt; \frac{n}{n+2} )</td>
<td>( k_T''(x) ) is positive. Stages get rapidly further from their design point with increasing axial distance through the compressor.</td>
</tr>
<tr>
<td>2</td>
<td>( \psi^* = \frac{n}{n+2} )</td>
<td>( k_T''(x) = 0 ). The distribution of ( k_T(x) ) is a straight line.</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{n}{n+2} &lt; \psi^* &lt; \frac{n}{n+1} )</td>
<td>( k_T''(x) ) is negative although ( k_T(x) ) does not tend toward an upper limit as ( x ) increases.</td>
</tr>
<tr>
<td>4</td>
<td>( \psi^* &gt; \frac{n}{n+1} )</td>
<td>( k_T''(x) ) is negative. As ( x ) increases, ( k_T(x) ) asymptotes toward ( Q = \frac{2k_U\psi^<em>(1-\psi^</em>)(k_M-k_U)}{\psi^<em>+n(1-\psi^</em>)} )</td>
</tr>
</tbody>
</table>

Fig 3.  \( k_T \) plotted against \( x \) for four values of \( \psi^* \).

The effect of this "repeating" phenomenon is to limit the movement of stage operating points from their design points: no matter how many stages in the compressor, the magnitudes of \( k_T, k_T'' \), \( k_T''(x) \) and \( k_T(x) \) never exceed certain fixed values.

The Effect of the Magnitude of \( \phi^* \). It is important to note that the analysis given above shows that \( \phi^*(x) \), the design distribution of flow coefficient, has no effect on the off-design performance of a compressor in the region of the design point. For straight-line \( \psi-\phi \) characteristics, as long as \( \psi^* \) is the same, a fixed percentage change in \( \phi \) from \( \phi^* \) will result in the same change in \( \psi \) from \( \psi^* \) and so the temperature and density distributions will change in the same way through the compressor, no matter what the original choice of \( \phi^*(x) \) at design. In practice this value of \( \phi^* \) may influence our ability to attain a certain value of \( \psi^* \).

The Form of the Overall Characteristics

Here we discuss the form of overall characteristics of the baseline compressor (considering only changes in speed and/or flow for a baseline compressor of constant design stage loading). We use the expression for the overall temperature rise (equation 16), rewriting it as follows for location II at exit from the compressor.
\[ \theta = \left( \frac{T_p/T_1}{T_p/T_1} \right) = 1 + kT_1I = 1 + \frac{Q}{P} \left[ 1 - \frac{1}{(rT^*)^P} \right] \] (30)

We wish to determine how the form of the overall characteristics (the constant speed lines on a plot of non-dimensional exit temperature, \( \theta \), against non-dimensional flow rate, \( M/M^* \)) varies with the two important parameters in equation 16 - the design stage loading coefficient (\( \psi^* \)) and the design overall temperature ratio (\( rT^* \)) (and therefore pressure ratio).

**The Slope of a Constant Speed Characteristic.** Differentiation of equation 30 at constant speed yields the slope of the non-dimensional constant speed line

\[ \theta' = \left( \frac{\partial \theta}{\partial (M/M^*)} \right)_U = \left( \frac{\partial \theta}{\partial kM} \right)_kU = \frac{P}{k} \left[ 1 - \frac{1}{\psi^*} \right] \left[ 1 - \frac{1}{(rT^*)^P} \right] \] (31)

which shows how the slope depends on \( \psi^* \) and \( rT^* \). This result would be modified if efficiency variations were taken into account. The slope of the characteristic is independent of the rotational speed (i.e. all constant speed lines have the same slope) for small perturbations in rotational speed about \( U^* \).

**Horizontal Spacing of Constant Speed Characteristics.** The horizontal spacing of the constant speed lines on the overall characteristic plot is given by

\[ M' = \left[ \frac{\partial (M/M^*)}{\partial (U/U^*)} \right]_{\theta} = \frac{\partial kM}{\partial kU} = \frac{1 + \psi^*}{1 - \psi^*} \] (32)

which is a function of the stage loading only.

The two parameters that we have considered, the design stage loading (\( \psi^* \)) and the design temperature ratio (\( rT^* \)) are linked by the number of stages in the compressor. A given duty may be achieved by a few stages of high loading or many stages of light loading. This relationship is expressed in the following equation

\[ rT^* = 1 + \frac{N \psi^* U^2}{C_p T_1} = 1 + \frac{N}{R} \] (33)

Figure 4 shows a typical set of "small perturbation" temperature - mass flow characteristics, for \( rT^* = 1.75 \) and \( \psi^* = 0.45 \). They form a set of straight lines of slope determined by \( rT^* \) and \( \psi^* \), and horizontal spacing determined by \( \psi^* \). If the overall temperature rise ratio is increased to 2.5 by keeping \( \psi^* \) unchanged and increasing the number of stages, the slope of the lines increases but their spacing is unchanged. The constant speed lines pivot about points with constant spacing as the overall temperature rise (and the number of stages of constant \( \psi^* \)) is increased, as shown in the figure. However, if we consider the characteristics of a compressor of a given duty (a fixed design temperature rise), then the slope of the constant speed lines depends on the value of the stage loading coefficient only. Figure 5 shows an example in which \( rT^* = 1.75 \); the magnitude of the slope can be seen to decrease rapidly with increasing \( \psi^* \). This figure illustrates a series of possible design choices; each value of \( \psi^* \) implies a value for the number of stages (or the speed of rotation) in accordance with equation 33.

Equations 31 and 32 were used to predict the shape of the overall characteristics for the Rolls-Royce Mamba compressor. Experimental characteristics together with the prediction are shown in figure 6. It can be seen that both the slope and spacing of the predicted characteristics agree reasonably well with the experimental results despite the simplifying assumptions used in the model.

**Working Line Considerations**

Figure 7a to 7d shows overall characteristics and stage characteristics for the front, middle and rear stages of a nine-stage compressor (\( \psi^* = 0.45 \), \( rT^* = 1.67 \)). The locations of nine operating points are shown on each characteristic and indicate how operating conditions on the overall characteristics are related to stage operating points. For the
The discussion of the previous section has shown that the magnitude of the stage loading coefficient ($\psi^*$) dominates the off-design performance of a compressor. We proceeded to investigate whether varying $\psi^*$ through the compressor, stage by stage, has a similarly powerful effect on the off-design performance.

In theory the axial flow compressor designer has a degree of flexibility in choosing the distribution of stage loading coefficient through the machine. In practice this flexibility is limited for three reasons. Firstly, to reduce weight and cost, there is usually a requirement to design a compressor with as few stages as possible, which, for a given duty, implies high stage loading. Secondly, it may be necessary to "off-load" certain stages if constraints on the annulus shape or features of the working line lead to certain stages being particularly susceptible to stall. Thirdly, the rotor speed in a stage is dependent on the radius of rotation of the rotor which itself may be constrained by the overall engine geometry.

If we remove the assumption that the design stage loading is constant through the compressor, and allow it to be a function of $x$, equation 13 for the temperature perturbation, $kT(x)$, must be rewritten in a more general form:

$$kT(x) = 
\frac{kM - kU - 1}{\frac{P}{R} \left[ 1 - \left( \frac{1 + x}{R} \right)^P \right]} \cdot \zeta(x)
$$

For $k\phi$ to be zero at an axial location $x = 1$, $kM = \frac{kU}{\zeta(1)}$, where $\zeta(x)$ is a function of $x$ that depends on the overall characteristics. An expression for $k\phi$ of such a line may be readily derived and is a function of $P$, $C$, $v^*$ and $r_T$. For the example quoted earlier ($T^* = 1.75$, $v^* = 0.45$) figure 8 shows such lines of "no movement" for the first, fifth and tenth stages of a ten stage compressor ($\sigma = 0,4,9$).

Most engine working lines will be quite close to the line shown as $\sigma = 4$, i.e., the middle stages will move relatively little from their design point - a phenomenon well known to compressor engineers. The additional point that we make here is that the slopes of the "no movement" lines for the entry and middle stages are relatively insensitive to the value of stage loading ($\psi^*$). However the slopes of these "no movement" lines increase with the overall design temperature ratio (or with the number of stages at a given loading), particularly at low values of temperature ratio. Thus for a compressor with $\psi^* = 0.45$ and $N$ stages, the slope of the line for the middle stage varies with different numbers of stages ($N$) and $T^*$ as shown in table 3.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$T^*$</th>
<th>Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.30</td>
<td>0.314</td>
</tr>
<tr>
<td>6</td>
<td>1.45</td>
<td>0.383</td>
</tr>
<tr>
<td>8</td>
<td>1.60</td>
<td>0.432</td>
</tr>
</tbody>
</table>

Table 3 The effect of changing design temp. ratio, for $\psi^* = 0.45$, on slope of "no movement" lines.
The Modelling of Design Changes to the Compressor

Equation 21, the most general solution for \( k_T(x) \), enables us to consider the effects of small changes to the design of the compressor on its off-design performance. Linear distributions were specified for the independent variables \( k_U, k_M, k_K \) and \( \lambda_A \): \( k_U = \lambda + \delta x, \ k_M = \varepsilon + \chi, \ k_K = \mu + vx \). We therefore have 8 parameters as inputs to the problem: \( \chi, \delta, \varepsilon, \mu, \lambda, \lambda_A \) and \( v \). Let us divide these parameters into 2 groups; in the first we put the “bulk” changes in speed and flow rate (\( \chi \) and \( \varepsilon \)) and in the second we put those parameters which require a redesign of the compressor for them to be implemented (\( \delta, \mu, \lambda, \lambda_A \) and \( v \)). We include in this second group \( \lambda, \lambda_A \) which describe changes in air injection/abstraction and stator stagger angle despite the fact that these adjustments are sometimes made to a compressor in service. The changes in a compressor brought about by “bulk” changes in speed and flow (\( \chi \) and \( \varepsilon \)) we refer to as “off-design” changes whereas changes in the other six parameters we refer to as “redesign” changes.

The design performance of the baseline compressor is denoted by state “B”, a redesigned compressor by state “D” and a superscript “•” indicates a change off-design. Then the change from B to B* is caused by non-zero values for \( \chi \) and/or \( \mu \) only (the first group of parameters) and the change from B to D involves non-zero values for \( \delta, \varepsilon, \mu, \lambda, \lambda_A \) and/or \( v \) only (the second group of parameters). The change from B to D* involves a non-zero value for at least one of the parameters from each of the 2 groups.

Because our analysis is based on a linearised model, the effects of moving from D to D* can be expressed as the effects of moving from B to D* minus the effects of moving from B to D (provided that the final state D* is itself a small perturbation from B) i.e., \( d(D \rightarrow D^*) = d(B \rightarrow D^*) - d(B \rightarrow D) \). We can therefore use equation 21 to investigate how changes to the design of a compressor can improve its off-design performance. By way of illustration we consider the effects of varying mean radius, using the expression for blade speed: \( k_U = \chi + \delta x \), and a “bulk” change in flow: \( k_M = \varepsilon \). We consider a non-zero value for \( \chi \) to be an off-design change as it could be facilitated by a simple change in rotational speed, but we consider a non-zero value for \( \delta \) as being a redesign change as it clearly implies a different compressor geometry (a positive value for \( \delta \) describes the mean radius increasing with axial distance through the compressor).

These effects are illustrated on an overall performance plot of \( k_T II \) against \( k_M \), in figure 9.

1) Consider first the design performance of the baseline compressor for which \( k_T II = k_U = k_M = 0 \). This is shown as point B on figure 9.

2) We now move the basic compressor off-design, from B to B*, by specifying \( k_U = \chi = 0.01 \) and \( k_M = \varepsilon = 0.01 \).
3) Consider next redesigning compressor B to have an increasing mean radius with axial distance. We substitute \( kU = \delta \chi \) (where \( \delta = 0.001 \)) into equation 22 and calculate \( kT \). Compressor D operates at point D on figure 9.

4) We now move compressor D off-design by specifying \( kU = \chi + \delta \chi \) and \( kM = \epsilon \). We arrive at point D'.

We could specify the magnitude and sign of \( \delta \) in accordance with a particular design objective e.g., to improve the position of an operating point from \( B \) to \( D' \) under a particular off-design condition (\( kU = \chi \) and \( kM = \epsilon \)). Similar changes in both design and off-design performance may be initiated by changes in \( kU, kK, kA \) and/or \( kA \), instead of \( kU \).

Note that there is a multitude of ways of selecting values for the input parameters to ensure that there any desired change, or no change, in compressor exit conditions from the design point. If we are free to select only one of the parameters \( kU, kM, kK, kA \) (and therefore two of the parameters \( \chi, \delta, \epsilon, \alpha, \kappa, \mu, \nu \)), then we are free to set the constants \( S \) and \( W \) in equation 21 to any desired value. We can therefore influence the distributions of \( kU, kM, kK, kA \) and \( kT \). To take a simple example, consider "bulk" changes in speed and flow (\( \chi \) and \( \epsilon \)) imposed on a compressor, but with a change in variable stator setting for all stages in the machine being possible (\( kU \) - the same change for each stage). Let us assume that this compressor was designed with an area perturbation of \( S = W = 0 \) and therefore \( kU \) is zero (and therefore \( W = 0 \) if we choose the stator stagger setting according to

\[
kA = \mu + \nu x
\]

where \( \mu = -\frac{\chi (1 + \psi^*)}{(1 - \psi^*)} \) and \( \nu = -\delta \frac{\chi (1 + \psi^*)}{(1 - \psi^*)} \).

With this change in area distribution \( S = W = 0 \) and so \( kT \), the change in compressor exit temperature, is zero; therefore our objective is met.

### The Effects of Errors in Blockage Estimates

The purposes of including area variation in the more general solution were twofold: firstly to enable the effects of changing the annulus area during the design process to be studied (as in the example above), and secondly to investigate the consequences of errors in annulus blockage estimates or changes in blockage between design and off-design conditions. By "blockage" we mean here the annulus area seemingly lost due to the displacement thicknesses of the endwall boundary layers.

Thus we first postulate a compressor, having been designed with an incorrect estimate of blockage, somehow running with this incorrect design level of blockage; in this condition all parameters are at their design values, denoted by an asterisk. We now imagine the blockage to change to its correct (actual) value and as a result the other parameters change to what we have previously referred to as "off-design" values.

The calculations outlined below were made for a 10 stage compressor.
compressor pressure ratio. The non-dimensional flow function at compressor inlet is unchanged and, as the compressor inlet pressure and temperature remain constant, the inlet mass-flow must remain the same; we therefore set \( K_M = 0 \). In this case, then, to obtain the solution we simply choose a value of \( k_T \) (a change in speed) such that the temperature at compressor exit equals its design value (i.e. \( k_T(10) = 0 \)). Figure 10b shows the solutions for \( k_T \) and \( k_0 \) plotted against \( x \). In this case the maximum change in flow coefficient from design is only -1.4\% and the small change in engine speed is -0.4\%. These are small values and are unlikely to cause serious problems to the compressor’s off-design operation. As a guideline, the change in flow coefficient is similar to the magnitude of the error in blockage, in percentage points. This analysis suggests that accurate estimation of blockage for the design of aero-engine compressors is less critical than has been previously thought.

CONCLUSIONS
1) An analysis of the off-design performance of multistage axial-flow compressors, based on an analytical solution, has been presented. This analytical solution is found to agree well with previous numerical solutions and provides more insight and understanding than current computer methods.
2) It is shown that, by changing the slopes of stage characteristics, variable stagger stator vanes have a major influence on the way that stage operating points move from their design points. The results of Goede and Casey (1988) have been obtained with an analytic solution.
3) It is shown that the mean design value of stage loading coefficient, \( \psi^* \), has a dominant influence on the off-design performance. It controls the spacing of constant speed overall characteristics and, together with the design temperature ratio, determines the slope of overall characteristics. The distribution of stage loading through the machine, \( \psi^*(x) \), is shown to have little effect. In the region of the design point, the magnitude and distribution of flow coefficient at design, \( \psi^* \), is seen to have little or no effect on compressor off-design performance.
4) For large values of \( \psi^* \) (and for compressors approaching stall), the stage operating points approach a “repeating stage” condition. At this condition, all stages operate at the same fixed percentages of design flow coefficient and loading.
5) The slope of the engine working line, overlaid on overall compressor characteristics, has a strong influence on the distribution of off-design movement through the compressor. For a typical working line it is shown that the middle stages of a compressor move little from their design operating points.
6) A method is presented for analysing the effects of design changes on an existing compressor. It is shown how judicious redesign can, within the limitations of small perturbations, counteract potential problems in off-design performance.
7) In the context of compressors for gas turbine engines, small errors in blockage estimates made during the design process do not have serious consequences for the installed compressor. This is due to the “self-adjusting” nature of the engine working line.

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