THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
345 E. 47th St., New York, N.Y. 10017

97-GT-274

THE TURBULENCE THAT MATTERS

BREAK

R.E. Mayle[1], K. Dullenkopf, and A. Schulz

Institut für Thermische Strömungsmaschinen
Universität Karlsruhe
Karlsruhe
Germany

ABSTRACT

A unified expression for the spectrum of turbulence is developed by asymptotically matching known expressions for small and large wave numbers, and a formula for the one-dimensional spectral function which depends on the turbulence Reynolds number Re_λ is provided. In addition, formulas relating all the length scales of turbulence are provided. These relations also depend on Reynolds number.

The effects of free-stream turbulence on laminar heat transfer and pre-transitional flow in gas turbines are re-examined in light of these new expressions using our recent thoughts on an 'effective' frequency of turbulence and an 'effective' turbulence level. The results of this are that the frequency most effective for laminar heat transfer is about $1.3U/L_e$, where U is the free-stream velocity and L_e is the length scale of the eddies containing the most turbulent energy, and the most effective frequency for producing pre-transitional boundary layer fluctuations is about $0.3U/\eta$ where η is Kolmogorov's length scale. In addition, the role of turbulence Reynolds number on stagnation heat transfer and transition is discussed, and new expressions to account for its effect are provided.

NOMENCLATURE

$E(k)$	three-dimensional spectral density, [m³/s²]
$E_1(k_1)$	one-dimensional spectral density, [m³/s²]
k	kinetic energy of the laminar fluctuations, [m²/s²]; three-dimensional wave number, [m⁻¹]
k_1	one-dimensional wave number, [m⁻¹]
k_d	wave number of energy dissipating eddies, [m⁻¹]
k_e	wave number of energy containing eddies, [m⁻¹]
L_e	length scale of the energy containing eddies, [m]
n	circular frequency, [Hz]
Re	Reynolds number
Re_λ	turbulence Reynolds number, $\sqrt{\overline{u'^2}}\lambda/\nu$
T	temperature, [°C]
Tu	free-stream turbulence level, $\sqrt{\overline{u'^2}}/U$
u	velocity component in the x-direction, [m/s]
U	free-stream velocity, [m/s]
v	velocity component in the y-direction, [m/s]
x	coordinate in the free-stream direction, [m]
y	coordinate normal to the surface, [m]

Greek

α	thermal diffusivity, [m²/s]; Kolmogorov's constant
δ	boundary layer thickness, [m]
ε	dissipation of kinetic energy, [m²/s³]
η	Kolmogorov's length scale, [m]
λ	viscous dissipation length scale, [m]
Λ	integral length scale of turbulence [m]
ν	kinematic viscosity, [m²/s]
ρ	density, [kg/m³]
υ	Kolmogorov's velocity scale, [m/s]

Additional Marks

$\overline{q}$	time-averaged component of q
q'	fluctuating component of q, $\overline{q'^2}=0$
$\sqrt{\overline{q'^2}}$	root-mean-square of q'

INTRODUCTION

The aerodynamic performance of a gas turbine and the heat load to its 'hot' surfaces are primarily determined by the free-stream pressure gradients and turbulence, as well as by the machine's pressure ratio and turbine inlet temperature. In general, the effects of the latter two are easily taken into account, while those of free-stream pressure gradients are well understood and can be calculated. The effects of free-stream turbulence, on the other hand, are now just beginning to be understood.

[1] ASME Fellow, Professor Emeritus of Mechanical Engineering, Rensselear Polytechnic Institute, Troy, NY, USA

Presented at the International Gas Turbine & Aeroengine Congress & Exhibition
Orlando, Florida — June 2–June 5, 1997
This paper has been accepted for publication in the Transactions of the ASME
Discussion of it will be accepted at ASME Headquarters until September 30, 1997

Two of its effects are clearly seen in Fig. 1 where the Nusselt number distributions around a typical fore-loaded airfoil for several free-stream turbulence levels are presented (Schulz, 1986). At the leading edge where the highest Nusselt numbers are found, and on the pressure surface ($x/c < 0$) where the highly accelerated boundary layer remains laminar, the main effect of turbulence is to cause a large increase in laminar heat transfer. On the suction surface ($x/c > 0$) where a laminar-to-turbulent transition occurs, the main effect is to cause an earlier onset of transition with its subsequent increase in turbulent heat transfer. Although these are not the only effects of free-stream turbulence in gas turbines, they are important effects and the only two we will presently consider.

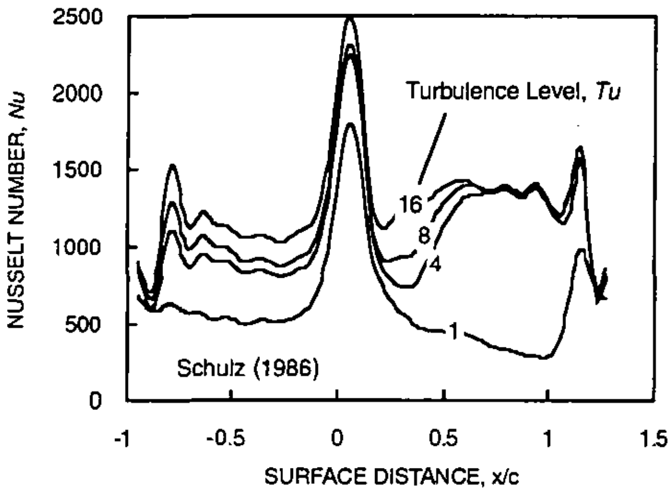

Figure 1. Effect of free-stream turbulence level on airfoil heat transfer

We have examined these effects before (Dullenkopf and Mayle, 1995, and Mayle and Schulz, 1996). In our 1995 paper, we focused our attention on how free-stream turbulence affects laminar heat transfer, and developed the concepts of an 'effective' frequency[2] and an 'effective' turbulence level. In the 1996 paper, we developed a laminar-kinetic-energy equation to calculate the effects of free-stream turbulence on pre-transitional laminar flow, and used the effective frequency-and-turbulence-level concepts to obtain the growth of laminar fluctuations in this flow. In both papers it is shown that the effects of free-stream turbulence are intimately connected to its spectral distribution, and not just its integrated level, or its level and integrated length scale, etc., but its full spectrum. In fact, we also showed that the effects of free-stream turbulence depend on the boundary layer thickness as well.

The present paper is our third in this series. Our intent is to examine the role of free-stream turbulence more closely and to present a more coherent picture of its effects than we did before. To accomplish this, we obtain a unified expression for the one-dimensional spectrum of turbulence which properly accounts for all the scales of turbulence, or in other words, the effect of turbulence Reynolds number.

[2] Depending on outlook, one may also substitute either the words 'effective wave number' or 'effective length scale' here.

The paper is divided into several parts. In the first part, we briefly review the effects of free-stream turbulence on stagnation-like heat transfer and pre-transitional non-accelerating laminar flow and discuss the turbulence that matters for each. In the second part, we develop the unified spectral distribution for turbulence and obtain formulas which can be used to predict the effects of free-stream turbulence on heat transfer and transition. Finally, in the last part of the paper, the important role of turbulence Reynolds number is discussed.

EFFECTS OF FREE-STREAM TURBULENCE

Effect on Heat Transfer

As it turns out (see Mayle and Schulz, 1996), a practical method of treating laminar boundary layers in turbulent free streams is that proposed by Lin (1957). Decomposing the velocities and pressure into time-averaged and time-dependent components, he obtained a set of equations very similar to Reynolds' equations for turbulent flow (Schlichting, 1979). The difference between these two sets of equations is the way in which the pressure fluctuations are handled. For laminar boundary layers in a turbulent free stream, the free-stream pressure fluctuations, as well as the mean pressure, are impressed on the flow in the boundary layer, while for turbulent flow they are not.

Following Lin's unsteady-laminar-flow analysis, the time-averaged energy equation for laminar boundary layers in turbulent free streams is

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} = \alpha \frac{\partial^2 \bar{T}}{\partial y^2} + \frac{\partial}{\partial y}\left(-\overline{v'T'}\right)$$

which is, not surprisingly, identical to that for turbulent flow. According to the usual notation, the over-bars refer to time-averaged quantities, u and v are the velocity components in the stream wise and wall-normal directions, x and y, respectively, T is the temperature, and α is the thermal diffusivity. In this case, however, the quantity $-\overline{v'T'}$ is a heat flux caused by the laminar fluctuations in the boundary layer arising from the turbulence in the free-stream.

For stagnation flow, Smith and Kuethe (1966) modeled this heat flux by considering the heat flux turbulent and using an eddy diffusivity proportional to the product of the free-stream turbulence intensity and the distance away from the wall. As a result, they predicted a linear increase in heat transfer with increasing free-stream turbulence level Tu, where $Tu = \sqrt{\overline{u'^2}}/U$, and $\sqrt{\overline{u'^2}}$ is the intensity of the free-stream velocity fluctuation in the direction of the mean velocity U. It is easy to show that the same result would be obtained if the heat flux was considered to be caused by laminar fluctuations. It wasn't until recently, however, after replacing Tu with an effective free-stream turbulence level Tu_{eff} that their predicted behavior was shown to correlate most of the data, including airfoil data and those for very high turbulence levels (Dullenkopf and Mayle, 1995). (For a consistent analysis of all the data, see Dullenkopf and Mayle, 1994.) The idea of introducing an effective turbulence level was based on the fact that only the turbulence within a small band of frequencies can really affect the boundary layer since any turbulence at higher frequencies will be viscously damped, while that at lower frequencies will appear quasi-steady and have no time-averaged effect.

Considering this last point in more detail, if $E_1(n)$ is the one-dimensional spectral density of turbulence where n is the frequency[3], and n_δ is a representative frequency of the band most affecting the boundary layer, it was shown by two of us (Dullenkopf and Mayle, 1995) that the effective free-stream turbulence level is given[4] by $Tu_{eff} = \sqrt{\overline{u'^2_{eff}}}/U \propto \sqrt{n_\delta E_1(n_\delta)}/U$. The proportionality factor, which depends directly on the square root of the bandwidth to frequency ratio, was supposed constant. In terms of the wave number $k_1 = 2\pi n/U$, and remembering that $E_1(n) = 2\pi E_1(k_1)/U$ (Hinze, 1975), the ratio of effective to integrated turbulence level Tu becomes

$$\frac{Tu_{eff}}{Tu} \propto \sqrt{\frac{k_{1\delta} E_1(k_{1\delta})}{\overline{u'^2}}} \tag{1}$$

where $k_{1\delta} = 2\pi n_\delta/U$ is the representative wave number of the effective wave number range. The 'best' value for $k_{1\delta}$, which was obtained by correlating stagnation heat transfer results (Dullenkopf and Mayle, 1995), is given by $k_{1\delta} \approx 0.11/\delta$.

The spectral function on the right hand side of eq. (1) is sketched in Fig. 2. The wave number k_e is that associated with the energy-containing eddies, hence the maximum near unity, or more exactly, as will be shown, at $k_{1max}/k_e \approx 1.3$. For large turbulence Reynolds numbers $k_e \approx 0.75\Lambda^{-1}$ where Λ is the integral length scale of turbulence. As an example, the effective turbulence level corresponding to a wave number ratio $k_{1\delta}/k_e \approx 0.15\Lambda/\delta$ with $\Lambda/\delta \approx 50$ is also shown. For larger values, corresponding to larger integral length scales or smaller boundary layer thicknesses, Tu_{eff} decreases; for smaller values, Tu_{eff} increases until the maximum effective turbulence level is obtained at $\Lambda/\delta \approx (k_{1max}/k_e)/0.15 \approx 9$, and then decreases. This is the effect of turbulence length scale on heat transfer previously described by us in 1995 (Dullenkopf and Mayle). It is also the effect described much earlier by Yardi and Sukhatme (1978) who measured the effects of turbulence level and length scale on stagnation heat transfer and found a maximum effect when $\Lambda/\delta \approx 10$. Therefore, the turbulence that matters for laminar heat transfer depends not on the measured (integrated) turbulence level, but on the turbulence level corresponding to a band of wave numbers around a wave number selected by the boundary layer, and the maximum effect is obtained for wave numbers in the energy containing range.

Effect on Transition

Transition in a laminar boundary layer begins at the first position along the surface where isolated spots of turbulence (Emmons, 1951) are formed, and as recently shown by two of us (Mayle and Schulz, 1996), the main effect of free-stream turbulence is to generate and amplify the laminar fluctuations in the boundary layer upstream of this position. Where these fluctuations attain a critical value, turbulent spots form and transition begins.

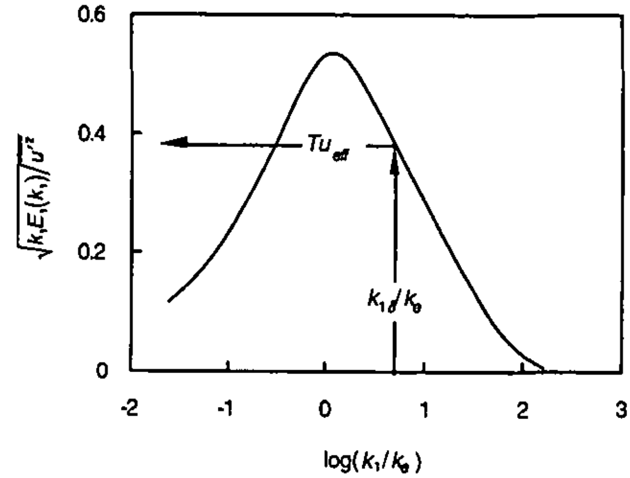

Figure 2. Spectrum function related to the effective turbulence for heat transfer in a laminar boundary layer

As we showed, the appropriate kinetic energy equation for these pre-transitional laminar fluctuations, called the laminar kinetic energy (LKE) equation, is

$$\bar{U}\frac{\partial k}{\partial x} + \bar{V}\frac{\partial k}{\partial y} = \overline{u'\frac{\partial u'_{fs}}{\partial t}} + \nu\frac{\partial^2 k}{\partial y^2} - \varepsilon$$

where k is the kinetic energy (not to be confused with the wave number), and ν is the kinematic viscosity. The terms on the right correspond respectively to the production, diffusion, and dissipation of laminar kinetic energy. The production term, shown as the time-average of the product of the velocity fluctuations in the boundary layer and the temporal derivative of those in the free-stream, is new and arises from taking the average of $u'(dp'/dx)$. Physically, it represents the work done on the fluctuations by the imposed fluctuating pressure forces. Since this term contains u'_{fs}, it provides a direct link between the free-stream turbulence and the fluctuations in the boundary layer. Therefore, these fluctuations are forced and do not arise from any natural instability in the flow.

From dimensional considerations it is easy to see that this term must be proportional to the product of $\sqrt{k}$, which is proportional to $\sqrt{\overline{u'^2}}$, an effective free-stream turbulence intensity $\sqrt{\overline{u'^2_{eff}}}$, as discussed above, and an effective frequency n_{eff}. Hence, the production term becomes proportional to $n_{eff}\sqrt{\overline{u'^2_{eff}}}\sqrt{k}$, of which only the quantity $n_{eff}\sqrt{\overline{u'^2_{eff}}}$ reflects the effects of free-stream turbulence. In our last paper (Mayle and Schulz, 1996), this quantity was identified as $C_\omega(U^2/\nu)\sqrt{\overline{u'^2_{fs}}}$ where C_ω is a dimensionless production rate. Replacing n_{eff} with an effective wave number and introducing the spectral density $E_1(k_1)$, using $\sqrt{\overline{u'^2_{eff}}} = Tu_{eff}U$ and eq. (1), one obtains

$$C_\omega \propto \frac{k_{1eff}\nu}{U}\sqrt{\frac{k_{1eff}E_1(k_{1eff})}{\overline{u'^2_{fs}}}} \tag{2}$$

Since the diffusion and dissipation of laminar kinetic energy are initially small, it is easy to show from the LKE equation, with a production term equal to $C_\omega(U^2/\nu)\sqrt{\overline{u'^2_{fs}}}\sqrt{k}$, that $\sqrt{k}$ increases in

[3] For readers refering to our previous papers, the following changes in nomenclature have been made: (D&M)–the circular frequency $f \rightarrow n$, the integral length scale $l \rightarrow \Lambda$; (M&S)–the circular instead of the angular frequency ω is used.

[4] Many of the results from our first two papers require some manipulation to obtain those stated in this paper. For the sake of brevity, the details have been omitted. The following result is an example. It is obtained by comparing the first three equations in the section titled 'A Simple Model' from D&M 1995.

proportion to C_ω. Therefore, the maximum growth of laminar fluctuations in the boundary layer, leading to the earliest onset of transition, occurs for the value of k_{1eff} which maximizes C_ω for a given free-stream turbulence level. The appropriate value of C_ω must then be given by

$$C_\omega \propto \left[\frac{k_1 \nu}{U}\sqrt{\frac{k_1 E_1(k_1)}{\overline{u'^2_{fs}}}}\right]_{max} \qquad (3)$$

The spectral function on the right hand side of eq. (3) is plotted in Fig. 3 against k_1/k_d, where k_d is the wave number associated with the eddies providing the largest contribution to the dissipation of turbulence. As will be shown, the maximum occurs at $k_1/k_d \approx 0.3$. This provides $k_{1eff} \approx 0.3k_d$, which is nearly the same result obtained previously by us using a slightly different approach (Mayle and Schulz, 1996). Therefore, the turbulence that matters for pre-transitional laminar flow depends not on the measured turbulence level, but on the turbulence level corresponding to a band of wave numbers near the dissipative range.

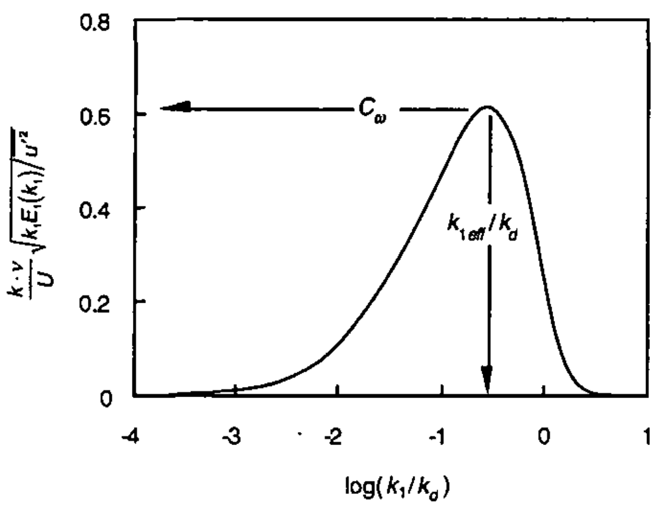

Figure 3. Spectrum function related to the production of laminar kinetic energy in a laminar boundary layer

<u>The Important Wave Numbers Ranges</u>

To highlight the important ranges of wave numbers involved, the two previous figures are combined in Fig. 4 using a common abscissa. As seen in this figure, the effective wave number for pre-transitional flow is more than one order of magnitude larger than the wave number which provides the maximum effect on laminar heat transfer. This magnitude, which depends on the turbulence Reynolds number, reflects the well known difference in wave numbers for the energy-containing eddies and those most responsible for dissipation. Using the values previously cited, their ratio becomes $k_{1eff}/k_{1max} = 0.25k_d/k_e$. For larger Reynolds numbers the ratio increases.

The curves in this figure have been drawn for a turbulence Reynolds number Re_λ of about 100, where $Re_\lambda = \sqrt{\overline{u'^2_{fs}}}\,\lambda/\nu$ and λ is Taylor's microscale of turbulence. Typical values for gas turbine engines are about 100 and above, which may be called moderate, with the smaller values found in the airfoils' wakes. These

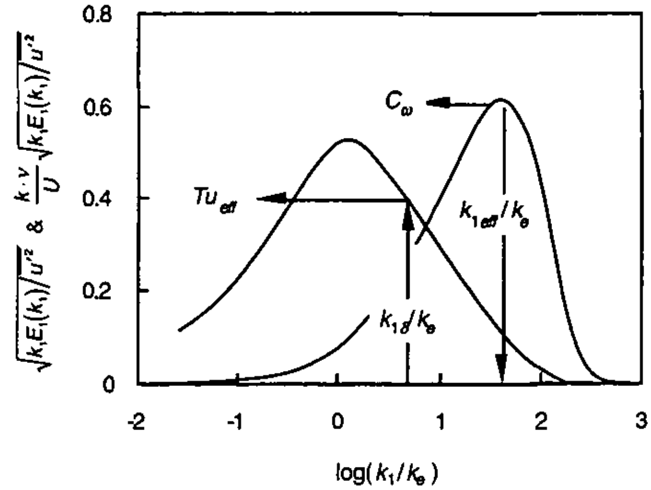

Figure 4. The different wave number ranges for the effects of turbulence on a laminar boundary layer

values can easily be attained in experiments using jet-grid-generated turbulence (Thole, 1992). Typical values in experiments with bar-grid-generated turbulence are about 50, which is low for gas turbine application. Large Reynolds numbers as referred to in the literature on turbulence correspond roughly to values > 2000.

A present difficulty, however, is that there is no easy way to account for the effects of turbulence Reynolds number on the spectrum of turbulence. In the following we attempt to provide one.

A UNIFIED SPECTRUM OF TURBULENCE

If an inertial subrange exists, where according to Kolmogorov the spectral density varies inversely as the wave number raised to the five-thirds power, it is possible to match von Kármán's (1948) interpolation formula for small wave numbers with Pao's (1965) theoretical expression for large wave numbers to provide a unified three-dimensional energy spectrum for isotropic turbulence. The formula given by von Kármán is

$$E(k) = 2^{17/6} E(k_e) \frac{(k/k_e)^4}{\left[1+(k/k_e)^2\right]^{17/6}} \qquad (4)$$

where k is the wave number (actually, the radius vector) in three-dimensional wave number space, and $E(k_e)$ is a function yet to be determined but which for very large turbulence Reynolds numbers is equal to $0.2\,\overline{u'^2}/k_e$.[5] For $k \ll k_e$, $E(k) \propto k^4$ which matches the behavior for very small wave numbers, while for $k \gg k_e$, $E(k) \approx 2^{17/6} E(k_e)(k/k_e)^{-5/3}$, which matches Kolmogorov's result for the inertial subrange.

For large wave numbers where viscous dissipation is important, the spectrum as determined by Pao is

$$E(k) = \alpha(\varepsilon\nu^5)^{1/4}(k/k_d)^{-5/3}\exp(-\tfrac{3}{2}r\alpha(k/k_d)^{4/3}) \qquad (5)$$

[5] Since free-stream turbulence is only being considered in this section, the subscript 'fs' on u' has been dropped.

where $\alpha \cong 1.7$ is Kolmogorov's constant[6], and ε is the rate of dissipation of turbulent kinetic energy. Originally taken as unity by Pao, the quantity r is introduced here to account for the effect of viscosity on the transfer of turbulent energy from one eddy to another and is, therefore, a function of Reynolds number. Since the dissipation of turbulent energy is given by

$$\varepsilon = 2\nu\int_0^\infty k^2 E(k)dk \tag{6}$$

the wave number k_d corresponds roughly to that where $k^2E(k)$ is maximum. It is also defined as $k_d = 1/\eta$ where $\eta = (\nu^3/\varepsilon)^{1/4}$ is Kolmogorov's length scale. For $k \ll k_d$, eq. (5) becomes $E(k) \approx \alpha(\varepsilon\nu^5)^{1/4}(k/k_d)^{-5/3}$, which again corresponds to the behavior for the inertial subrange.

Supposing that an inertial subrange exists, say between k^* and k^{**}, eqs. (4) and (5) must overlap when $k_e \ll k^* < k^{**} \ll k_d$. Comparing eqs. (4) and (5) in this region provides $2^{17/6}E(k_e) = \alpha(\varepsilon\nu^5)^{1/4}(k_d/k_e)^{5/3}$. Neglecting the error of eq.(4) at k^* and that of eq. (5) at k^{**}, which tacitly implies large turbulence Reynolds numbers, eqs. (4) and (5) may be combined into one unified expression for all wave numbers, namely

$$E(k) = \alpha(\varepsilon\nu^5)^{1/4}(k_d/k_e)^{5/3}\frac{(k/k_e)^4\exp\left[-\frac{3}{2}r\alpha(k/k_d)^{4/3}\right]}{\left[1+(k/k_e)^2\right]^{17/6}} \tag{7}$$

This expression is plotted in Fig. 5 for k_e/k_d=0.002 (corresponding to $Re_\lambda \approx 450$). In addition, Kolmogorov's relation and inertial subrange are shown. For $k_e/k_d > 0.01$, corresponding to $Re_\lambda < 100$, the curve begins to fall below Kolmogorov's relation and less, if any, of it can be said to exhibit an inertial subrange. In spite of this, and remembering that eq. (7) remains completely valid when an inertial subrange does exist, we propose that it may also be an appropriate interpolation formula when a subrange does not exist. Of course, this remains to be seen.

Using the above expression for $E(k)$, the one-dimensional energy spectrum of turbulence can be obtained from

$$E_1(k_1) = \int_{k_1}^\infty \frac{E(k)}{k}\left(1-\frac{k_1^2}{k^2}\right)dk \tag{8}$$

In addition, the dissipation can be obtained from eq. (6), and the integrated mean-square velocity and the integral length scale can be obtained from

$$\overline{u'^2} = \int_0^\infty E_1(k_1)dk_1 \quad \text{and} \quad \Lambda = \frac{\pi}{2\overline{u'^2}}\int_0^\infty \frac{E(k)}{k}dk, \tag{9}$$

respectively. Once ε and $\overline{u'^2}$ have been evaluated, the relation between the dissipation and the work done by the energy-containing eddies, namely, $\varepsilon = A(\overline{u'^2})^{3/2}k_e$, may be evaluated to determine $A = \text{fnc}(Re_\lambda)$. The results of these calculations are given in the Appendix where tables of the various integrals and turbulence parameters are presented for Re_λ = 26 to 4980. The one-dimensional spectrum, however, was calculated only for Re_λ = 34, 106, and 238.

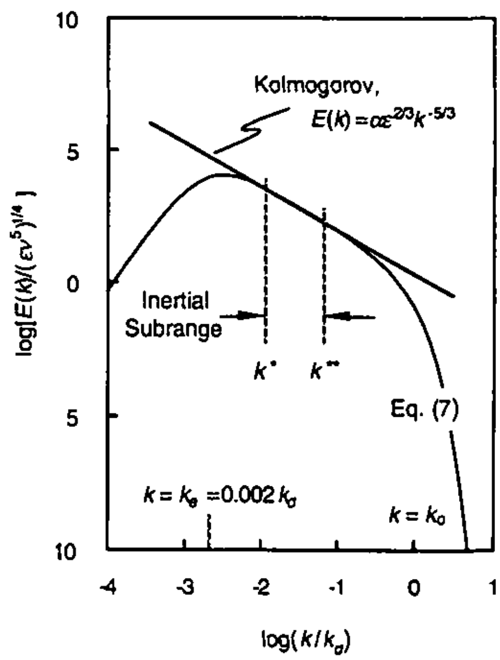

Figure 5. A unified three-dimensional energy spectrum of turbulence (α = 1.7)

The results for $E_1(k_1)$ normalized using the dissipative scales of turbulence are plotted in Fig. 6. Pao's result for infinitely large Reynolds numbers is also shown. In this format, the Reynolds number independence of the energy-dissipating eddies (large k_1) is clearly seen. In addition, the increasing amount of energy contained in the large eddies (small k_1) with increasing Reynolds number is also seen. The energy density of the large eddies can be calculated[7], and is shown in Fig. 7 together with data from experiments on grid turbulence and boundary layers compiled by Chapman (1979). The agreement is good even for small Re_λ, and suggests that the proposed expression for $E(k)$ may be useful even when an inertial subrange does not exist.

Plots of $E_1(k_1)$ normalized using the integral scales of turbulence are presented in Fig. 8. In this format, the viscous effect on the energy-dissipating eddies is easily seen. The result for infinitely large Reynolds numbers is due to von Kármán, namely, $2\pi E_1(k_1)/\overline{u'^2}\Lambda = 4[1+(k_1/k_e)^2]^{-5/6}$, which is obtained by integrating eq. (4). According to this expression, it is natural to use Λ rather than $1/k_e$ as the length scale for the ordinate and $1/k_e$ for the abscissa. For large Reynolds numbers, Λ and $1/k_e$ may be freely interchanged as commonly done since $\Lambda k_e \approx 0.74$, a constant. For lower Reynolds numbers, however, the product varies[8] approximately as $0.74(1+90/Re_\lambda)^{2/9}$, which produces an increasing rightward shift in the curves of Fig. 8 if Λk_1 rather than k_1/k_e is used for the abscissa. This shift actually produces an overlapping of spectral curves for different Re_λ near $k_1/k_e = 1$, which is commonly found in data plotted this way and worthy to note.

[6] An average value for a large variety of flows as obtained by Sreenivasan (1995) is α=1.62±0.17. The value of 1.7 is that found by Pao (1965).

[7] Noting that the integral for $E_1(k_1)$ with $k_1 = 0$ is identical to the integral for Λ (see eqs. 8 and 9), the energy density for these eddies as a function of Re_λ can be determined from values found in Tables A1 and A2 of the Appendix.

[8] Obtained by a curve fit to the calculated results in Tables A1 and A2.

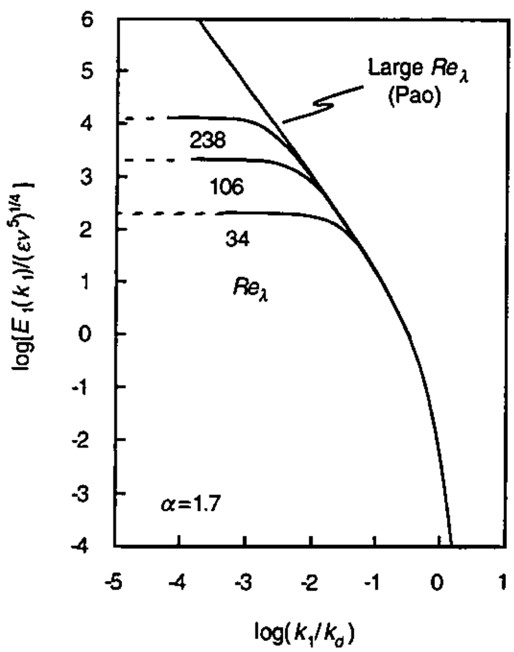

Figure 6. One-dimensional energy spectrum using the dissipative scales of turbulence

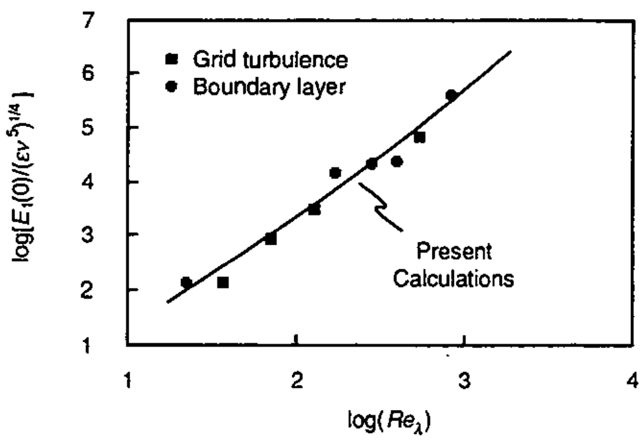

Figure 7. Calculated values of the energy density at $k_1 = 0$ compared with experimental results

Spectral Function Formulas

Formulas for the spectral function and various other useful relations were obtained by curve fits to the calculated results in Appendix A. As already mentioned above the relation between Λ and k_e is approximately given by

$$\Lambda k_e = 0.74(1+90/Re_\lambda)^{2/9} \tag{10}$$

The relations between all the other length scales depend on A. Using the results in Table A2 of the Appendix, a good approximation for A is given by

$$A = 0.80(1+24/Re_\lambda) \tag{11}$$

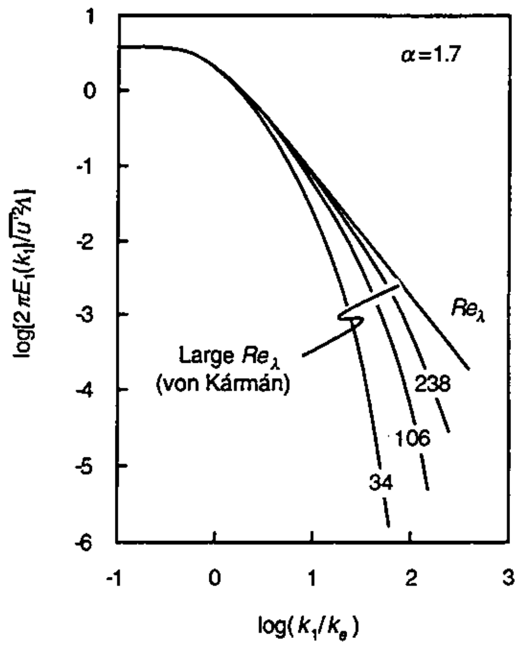

Figure 8. One-dimensional energy spectrum using the integral scales of turbulence

Then the relation between the energy-containing and energy-dissipating eddy scales becomes

$$\frac{k_d}{k_e} = 15^{-3/4} A Re_\lambda^{3/2} = 0.105(1+24/Re_\lambda)Re_\lambda^{3/2} \tag{12}$$

Pao's result for large Re_λ (shown in Fig. 6) is given with good accuracy up to the third moment with respect to k_1 by

$$\frac{E_1(k_1)}{(\epsilon\nu^5)^{1/4}} = \tfrac{18}{55}\alpha(k_1/k_d)^{-5/3}\exp\left[-\tfrac{14}{3}(k_1/k_d)\right]; \text{ (large } Re_\lambda) \tag{13}$$

As noted before, von Kármán's one-dimensional spectral distribution is given by

$$\frac{2\pi E_1(k_1)}{\overline{u'^2}\Lambda} = \frac{4}{\left[1+(k_1/k_e)^2\right]^{5/6}}; \text{ (large } Re_\lambda) \tag{14}$$

An expression for the complete range of Reynolds numbers, which is also good up to the third moment of accuracy, is

$$\frac{E_1(k_1)}{(\epsilon\nu^5)^{1/4}} = \tfrac{18}{55}\alpha\frac{r(k_d/k_e)^{5/3}}{1+r(k_1/k_e)^{5/3}\exp\left[\tfrac{14}{3}(k_1/k_d)\right]} \tag{15}$$

where a good approximation for r (obtained by fitting the results in Tables A1 and A2 of the Appendix) is

$$r = 1-15Re_\lambda^{-3/2} \tag{16}$$

An equivalent expression for $E_1(k_1)$ is

$$\frac{2\pi E_1(k_1)}{\overline{u'^2}\Lambda} = \frac{4}{1+r(k_1/k_e)^{5/3}\exp\left[\tfrac{14}{3}(k_1/k_d)\right]} \tag{17}$$

Since r approaches unity for large Re_λ, it is readily seen that these expressions reduce to eqs. (13) and (14) for both small and large wave numbers in the limit of large Reynolds number.

A comparison of eq. (17) with data for bar-grid-generated isotropic turbulence is shown in Fig. 9. To make the comparison, eqs. (10), (12) and (16) were also used. The agreement is seen to be very good.

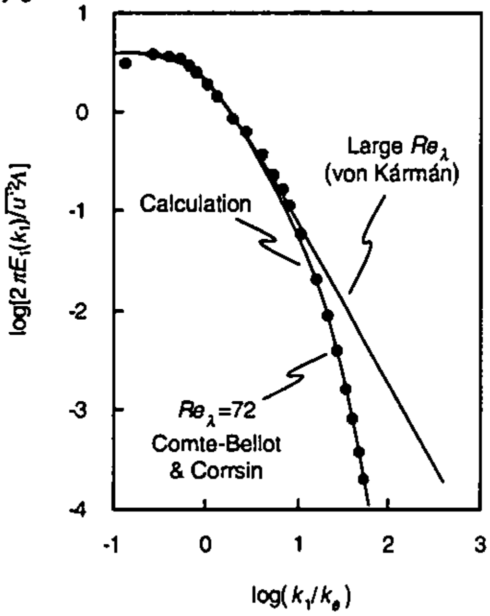

Figure 9. Calculated and experimental one-dimensional energy spectrums of turbulence

THE ROLE OF TURBULENCE REYNOLDS NUMBER

The effect of Reynolds number on Tu_{eff}/Tu is shown in Fig. 10 where its spectral function for Re_λ = 34, 106, and 238 are plotted. The curve for Re_λ = 106 was already used in Fig. 2. As previously mentioned, it corresponds to the lower limit for gas turbine engines and the higher limit found in most bar-grid-generated turbulence experiments.

Clearly, the effect of Re_λ is less pronounced for higher values, where viscous effects are less important, than for the lower. Therefore, once Re_λ is larger than about 200, the results we previously presented (Dullenkopf and Mayle, 1995), which were based on von Kármán's distribution, may be used. Using the spectral distribution given by eq. (17) instead of von Kármán's, eq. (6)[*] of that paper should be replaced with

$$\frac{Tu_{eff}}{Tu} = 1.29\sqrt{\frac{\Lambda_s(\Lambda k_e)}{1+0.01r\Lambda_s^{5/3}\exp\left[0.29\Lambda_s(k_e/k_d)\right]}} \quad (18)$$

where $\Lambda_s = \Lambda\sqrt{(a/v)}$, and $a = dU/dx$ is the free-stream strain rate.

The effect of Reynolds number on C_ω is shown in Fig. 11. Clearly, the value of C_ω is greatly affected by Re_λ which underlines the importance of the free-stream turbulence Reynolds number on the production of LKE in pre-transitional flow and, consequently, on the onset of transition.

Using eq. (13), it is easy to show that the maxima occur at $k_{1eff}/k_d = 0.3$. Substituting this value into eq. (2) together with $k_d = (\varepsilon/v^3)^{1/4}$ yields $C_\omega \propto (\varepsilon v)^{1/2}/U\sqrt{\overline{u_{fs}'^2}}$. By using the various relations between turbulence length and velocity scales (Hinze), many different expressions may be obtained for C_ω. The simplest is $C_\omega \propto v/U\lambda$, which is also easy to evaluate from turbulence decay measurements since $d(\ln Tu)/dx = -5(v/U\lambda)^2$. Evaluating the constant of proportionality using the information presented in Tables 1 and 2 of our previous paper (Mayle and Schulz, 1996), we find

$$C_\omega = 0.27\frac{v}{U\lambda} \quad (19)$$

In our previous paper, we had obtained $C_\omega = 0.07(\upsilon/U)^{2/3}(v/U\Lambda)^{1/3}$ where $\upsilon = (\varepsilon v)^{1/4}$ is Kolmogorov's velocity scale. After some rearrangement this becomes $C_\omega = 0.30v/U\lambda$, which is nearly the same as that above. Presently, however, we suggest that eq. (19) be used for the coefficient of the production term in the LKE equation (see subsection above titled "Effect on Transition"). This leads to a new form for the LKE equation, namely

$$\bar{u}\frac{\partial k}{\partial x} + \bar{v}\frac{\partial k}{\partial y} = 0.085\sqrt{\frac{\varepsilon_\infty}{v}}U\sqrt{k}e^{-y^+/13} + v\frac{\partial^2 k}{\partial y^2} - \varepsilon \quad (20)$$

where $y^+ = yu_\tau/v$, $\varepsilon = 2vk/y^2$ (see eq. 6, Mayle and Schulz, 1996), and u_τ is the shear velocity.

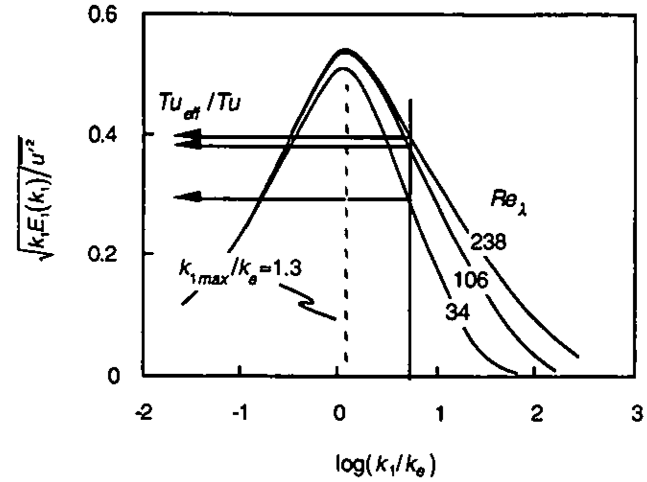

Figure 10. Effect of Reynolds number on the effective turbulence for heat transfer in a laminar boundary layer

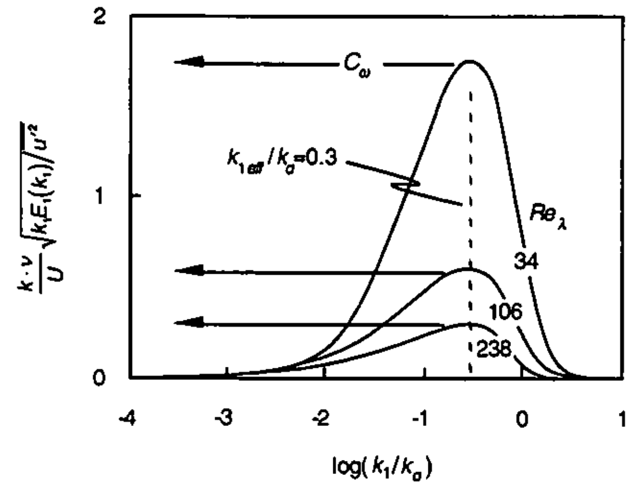

Figure 11. Effect of Reynolds number on the production of laminar kinetic energy in a laminar boundary layer

[*] Note that the left-hand-side of this equation is equal to Tu_{eff}/Tu.

CONCLUSIONS

In this paper, the third in our series concerned with the effects of free-stream turbulence in turbomachines, we have concentrated on the turbulence itself and developed a unified expression for the turbulence spectrum which can be used, together with the results of our two previous papers, to explain the effects of turbulence on laminar heat transfer and transition from laminar to turbulent flow. The main result of all three papers is that the effects of turbulence can only be explained when the full spectrum of turbulence is known such that the relevant 'effective' turbulence levels and frequencies can be determined.

For the complex flows found in turbomachinery, however, this requirement appears almost impossible to satisfy. But from an engineering standpoint, the results of this paper provide a good idea about the turbulence that matters for both laminar heat transfer and transition, and indicate that perhaps the only other quantity needed besides the free-stream turbulence level, if the spectral distribution developed in this paper is accepted, is the turbulence Reynolds number Re_λ (or a Reynolds number based on another turbulence length scale). This conclusion is obtained from the observation that all length scales are related through Re_λ, and if both Tu and Re_λ are known, then λ and all the length scales are known. Of course, the spectral distribution presented in this paper may not apply, but so far we have nothing else.

For the gas turbine designer, the integral length scale of turbulence Λ, rather than the microscale λ, is easier to estimate. In a free stream unaffected by wakes, Λ will be of the same order as the gap in the upstream airfoil row. In the wakes, Λ will be of the same order as the upstream airfoil's trailing edge. From these estimates, a reasonable range of turbulence Reynolds numbers can be obtained and used, together with the results presented in this paper, to project the effects of free-stream turbulence on both heat transfer and aerodynamic performance.

ACKNOWLEDGMENTS

The lead author is indebted to Professor S. Wittig for his continuing support. The same author is also grateful to the Sonderforschungsbereich 167 which funded most this work during the author's 1996 summer stay as a Visiting Professor at the Institut für Thermische Strömungsmaschinen.

REFERENCES

Batchelor, G.K., 1953, *The Theory of Homogeneous Turbulence*, Cambridge University Press.

Chapman, D.R., 1979, "Computational Aerodynamics Development and Outlook," AIAA Journal, 17, pp. 1293–1313.

Dullenkopf, K., and Mayle, R.E., 1994, "The Effects of Incident Turbulence and Moving Wakes on Laminar Heat Transfer in Gas Turbines," ASME JOURNAL OF TURBOMACHINERY, 116, pp. 23–28.

Dullenkopf, K., and Mayle, R.E., 1995, "An Account of Free-Stream-Turbulence Length Scale on Laminar Heat Transfer," ASME JOURNAL OF TURBOMACHINERY, 117, pp. 401–406.

Emmons, H.W., 1951, "The Laminar-Turbulent Transition in a Boundary Layer – Part I," *J. Aero. Sci.*, 18, pp. 490–498.

Hinze, J.O., 1975, *Turbulence*, McGraw-Hill.

Kármán, T. von, 1948, "Progress in the Statistical Theory of Turbulence," *Proc. Natl. Acad. Science*, 34, p. 530.

Lin, C.C., 1957, "Motion in the Boundary Layer with a Rapidly Oscillating External Flow," *Proc. 9th Int. Congress Appl. Mech.*, Brussels, 4, pp. 155–167.

Mayle, R.E., and Schulz, A., 1996, "The Path to Predicting Bypass Transition," ASME Paper No. 96-GT-199, (to be published in the JOURNAL OF TURBOMACHINERY).

Pao, Y.H., 1965, "Structure of Turbulent Velocity and Scalar Fields at Large Wavenumbers," Phys. Fluids, 8, pp. 1063–1075.

Schlichting, H., 1979, *Boundary-Layer Theory*, McGraw-Hill, New York.

Schulz, A., 1986, " Zum Einfluß hoher Freistromturbulenz, intensiver Kühlung und einer Nachlaufströmung auf den äußeren Wärmeübergang einer konvektiv gekülten Gasturbinenschaufel," Ph.D. Thesis, U. Karlsruhe, Karlsruhe, Germany (also Wittig, S., Schulz, A., Bauer, H.J., and Sill, K.H., 1985, "Effects of Wakes on the Heat Transfer in Gas Turbine Cascades," AGARD CP 390, pp. 6-1–6-13).

Smith, M.C., and Kuethe, A.M., 1966, "Effects of Turbulence on Laminar Skin Friction and Heat Transfer," *Physics of Fluids*, 9, pp. 2337–2344.

Sreenivasan, K.R., 1995, "On the Universality of the Kolmogorov Constant," *Physics of Fluids*, 7, pp. 2778–2784.

Thole, K.A., 1992, "High Freestream Turbulence Effects on the Transport of Heat and Momentum," Ph.D. Thesis, University of Texas at Austin.

Yardi, N.R., and Sukhatme, S.P., 1978, "Effect of Turbulence Intensity and Integral Length Scale of a Turbulent Free Stream on Forced Convection Heat Transfer from a Circular Cylinder in Crossflow," *Proc. 6th Int. Heat Transfer Conference*, Toronto, Canada, Vol. 5, FC(b)-29, pp. 347–352.

APPENDIX

With the three-dimensional energy spectral density for isotropic turbulence given as

$$E(k) = \alpha(\varepsilon\nu^5)^{1/4}(k_d/k_e)^{5/3}\frac{(k/k_e)^4 \exp\left[-\tfrac{3}{2}r\alpha(k/k_d)^{4/3}\right]}{\left[1+(k/k_e)^2\right]^{17/6}}$$

eqs. (6), (8), and (9) in the text may be expressed respectively as

$$\alpha = (k_d/k_e)^{4/3} I_\varepsilon^{-1}$$

$$\frac{E_1(k_1)}{(\varepsilon\nu^5)^{1/4}} = \tfrac{1}{2}\alpha(k_d/k_e)^{5/3} I_E = \tfrac{1}{2}(k_d/k_e)^3 I_E / I_\varepsilon$$

$$\frac{\overline{u'^2}}{(\varepsilon\nu^5)^{1/4}k_e} = \tfrac{1}{3}\alpha(k_d/k_e)^{5/3} I_u = \tfrac{1}{3}(k_d/k_e)^3 I_u / I_\varepsilon$$

$$\frac{\overline{u'^2}\Lambda}{(\varepsilon\nu^5)^{1/4}} = \tfrac{1}{4}\pi\alpha(k_d/k_e)^{5/3} I_\Lambda = \tfrac{1}{4}\pi(k_d/k_e)^3 I_\Lambda / I_\varepsilon$$

where

$$I_\varepsilon = \int_0^\infty \frac{z^{5/2}\exp(-bz^{2/3})}{(1+z)^{17/6}}\,dz,$$

$$I_E = (k_1/k_e)^4 \int_0^\infty \frac{(z-1)\exp\left[-b(k_1/k_e)^{4/3}z^{2/3}\right]}{\left[1+(k_1/k_e)^2 z\right]^{17/6}}\,dz,$$

$$I_u = \int_0^\infty \frac{z^{3/2}\exp\left(-bz^{2/3}\right)}{(1+z)^{17/6}}dz, \quad I_\Lambda = \int_0^\infty \frac{z\exp\left(-bz^{2/3}\right)}{(1+z)^{17/6}}dz,$$

$$\text{and} \quad b = \tfrac{3}{2} r\alpha (k_e/k_d)^{4/3} = \tfrac{3}{2} r I_\varepsilon^{-1}$$

The integrals I_ε, I_u, and I_Λ have been evaluated for several values of b and the results tabulated in Table A1. The frequency parameter $n\Lambda/U = (k_1/k_e)(\Lambda k_e/2\pi)$ is often used in spectral plots, so the quantity $\Lambda k_e/2\pi = 3I_\Lambda/8I_u$ is also provided. Note for $b \to 0$, corresponding to large Reynolds numbers, $\Lambda k_e \to 0.75$ as expected. It should also be noted that the results presented in Table A1 are independent of Kolmogorov's constant α.

Table A1
Turbulence Spectrum Integrals

b	I_ε	I_u/I_ε	I_Λ/I_ε	r	$\Lambda k_e/2\pi$
3.0E-2	4.45E+1	2.93E-2	1.30E-2	0.890	0.1671
2.0E-2	6.91E+1	2.05E-2	6.70E-3	0.921	0.1587
7.5E-3	1.94E+2	8.48E-3	3.25E-3	0.970	0.1436
3.4E-3	4.34E+2	4.09E-3	1.48E-3	0.984	0.1359
2.0E-3	7.43E+2	2.47E-3	8.71E-4	0.991	0.1320
6.0E-4	1.87E+3	1.03E-3	3.49E-4	0.996	0.1272
5.0E-4	2.99E+3	6.52E-4	2.18E-4	0.998	0.1256
1.0E-4	1.50E+4	1.34E-4	4.36E-5	0.999	0.1219
3.0E-5	5.00E+4	4.07E-5	1.31E-5	1.000	0.1205

Once α is chosen, the turbulence parameters $k_d/k_e = (\alpha I_\varepsilon)^{3/4}$, $A = (3/\alpha I_u)^{3/2}$, and $Re_\lambda = (15^{1/2}/3)\alpha^{3/2} I_\varepsilon^{1/2} I_u$ can be determined. These quantities have been evaluated using $\alpha = 1.7$ and the results provided in Table A2. The quantity $(A\Lambda k_e)^{-1}$, which is the ratio of the dissipation and integral length scales, is also provided. Experimental values for this quantity typically lie between 1.1 and 1.5. Batchelor (1953) quotes values between 0.8 and 1.3.

The integral $I_E(k_1)$ was evaluated for b=0.02, 0.0034, and 0.0008, which for α=1.7 correspond to Re_λ=34, 106, and 237, respectively. The results of these calculations are presented in Table A3. Since the one-dimensional frequency spectrum $E_1(n)$ is often used, the quantity $UE_1(n)/\overline{u'^2}\Lambda = 2\pi E_1(k_1)/\overline{u'^2}\Lambda = 4I_E/I_\Lambda$ is also given in the table. Note that for $k_1/k_e \to 0$, corresponding to low frequencies, $UE_1(n)/u'^2\Lambda \to 4$ as expected. Except for Re_λ, the values given in this table are independent of the value assigned to α. For those interested in performing other calculations, replacing the values of Re_λ with their corresponding values of b from Table A2 makes the whole table independent of α.

Table A2
Turbulence Parameters ($\alpha = 1.7$)

b	k_d/k_e	A	Re_λ	$(A\Lambda k_e)^{-1}$
3.0E-2	26	1.58	25	0.60
2.0E-2	36	1.39	34	0.72
7.5E-3	77	1.11	66	1.00
3.4E-3	142	0.992	106	1.18
2.0E-3	212	0.941	143	1.28
8.0E-4	423	0.882	237	1.42
5.0E-4	602	0.861	305	1.47
1.0E-4	2020	0.821	705	1.59
3.0E-5	4980	0.807	1300	1.64

Table A3
One-Dimensional Energy Spectrum

	$Re_\lambda = 34$		$Re_\lambda = 106$		$Re_\lambda = 238$	
k_1/k_e	I_E/I_ε	$4I_E/I_\Lambda$	I_E/I_ε	$4I_E/I_\Lambda$	I_E/I_ε	$4I_E/I_\Lambda$
0.0251	8.69E-3	3.99	1.48E-3	4.00	3.49E-4	4.00
0.0398	8.68E-3	3.99	1.48E-3	3.99	3.48E-4	4.00
0.0631	8.66E-3	3.98	1.48E-3	3.99	3.48E-4	3.99
0.100	8.62E-3	3.96	1.47E-3	3.97	3.46E-4	3.97
0.158	8.50E-3	3.91	1.45E-3	3.92	3.42E-4	3.92
0.251	8.23E-3	3.78	1.41E-3	3.80	3.31E-4	3.80
0.398	7.62E-3	3.50	1.31E-3	3.53	3.08E-4	3.54
0.631	6.43E-3	2.96	1.12E-3	3.01	2.63E-4	3.02
1.00	4.65E-3	2.14	6.23E-4	2.22	1.95E-4	2.24
1.58	2.75E-3	1.27	5.08E-4	1.37	1.22E-4	1.40
2.51	1.35E-3	6.23E-1	2.70E-4	7.26E-1	6.56E-5	7.53E-1
3.98	5.70E-4	2.62E-1	1.29E-4	3.48E-1	3.23E-5	3.71E-1
6.31	2.07E-4	9.52E-2	5.76E-5	1.56E-1	1.51E-5	1.74E-1
10.0	6.21E-5	2.86E-2	2.42E-5	6.53E-2	6.68E-6	7.90E-2
15.8	1.42E-5	6.52E-3	9.38E-6	2.53E-2	3.03E-6	3.48E-2
25.1	2.10E-6	9.68E-4	3.24E-6	8.75E-3	1.28E-6	1.47E-2
39.6	1.53E-7	7.01E-5	9.38E-7	2.53E-3	5.08E-7	5.83E-3
63.1	3.36E-9	1.54E-6	2.04E-7	5.51E-4	1.84E-7	2.11E-3
100	9.12E-12	4.19E-9	2.83E-6	7.65E-5	5.71E-8	6.55E-4
158	–	–	1.85E-9	5.00E-6	1.40E-8	1.60E-4
251	–	–	–	–	2.33E-9	2.68E-5