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GREEN'S FUNCTIONS FOR TIMOSHENKO BEAM PROBLEMS

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ABSTRACT: In this paper, a unified formulation is given for the bending, buckling and vibration problems of uniform Timoshenko and Euler-Bernoulli beams resting on various models of elastic foundation. Canonical Green's functions have been derived for these beams which can be readily used to furnish exact solutions. In addition to elucidating the behaviour of the beams, the exact solutions serve as important benchmark results for checking the convergence and accuracy of various solutions obtained from numerical methods.

INTRODUCTION

Recently, Lueschen *et al.* (1996) derived closed form expressions for the Green's functions of uniform Timoshenko and Euler-Bernoulli beams. Both bending and vibration beam problems were addressed.

Motivated by the work of Lueschen *et al.* (1996), this paper generalizes their Green's functions for the bending, buckling and vibration problems of Timoshenko and Euler-Bernoulli beams resting on various models of elastic foundation. All six combinations of the classical end conditions are considered. The present paper complements many papers that have been written on this subject (Lee *et al.* 1992, Wang and Stephens 1977, Naidu and Rao 1995, Shirima and Giger 1992, Rosa 1995 and Razaqpor and Shah 1991), by giving a comprehensive and unified treatment of the aforementioned beam problems and by providing exact solutions in a canonical form. It is hoped that the paper will serve as a useful reference source of exact beam solutions to researchers and academicians working on Timoshenko beam problems that are of fundamental and practical importance.

PROBLEM FORMULATION

Consider an elastic beam of length *L*, cross-sectional area *A*, second moment of area *I*, mass density ρ, modulus of elasticity *E*, and shear modulus *G*. The beam is subjected to a transverse distributed load $q(x)$, n point loads P_i, $(i = 1,2,\ldots,n)$ located at $x = e_i$ and an axial tensile preload *N*. The beam rests on an elastic foundation as shown in Fig. 1. The various models of elastic foundation considered are described below:

- Winkler foundation (Lee *et al* 1992) having a modulus K_w.
- Pasternak foundation (Wang and Stephens 1977 and Naidu and Rao 1995) having the foundation moduli K_w and K_s. The second foundation parameter K_s is the stiffness of the shearing layer. This model assumes that there is a shear interaction between the springs, and the top ends of the springs are connected to an incompressible layer which resists only transverse shear deformation.
- Generalized foundation (Shirima and Giger 1992 and Rosa 1995) having the foundation moduli K_w and K_M. This model assumes that at the point of contact between the beam and the foundation, there are both pressure forces and moments. This model has two versions; one version assumes the bending moment to be proportional to the bending rotation of the beam while the other version assumes the bending moment to be proportional to the total rotation. The proportionality constant is K_M.
- Vlasov foundation (Razaqpur and Shah 1991) having the foundation moduli K_w and K_v. The foundation is treated as a semi-infinite medium. The second foundation parameter is defined as

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$$K_V = \frac{E_S}{4(1+\nu_S)}\frac{B}{\mu} \quad (1)$$

where E_s is the Young's modulus of foundation, ν_s the Poisson's ratio of foundation and B the width of the beam. The parameter μ characterizes the rate at which vertical deformation of foundation decays with depth and it can be correlated to the beam displacements or it can be determined by means of an iterative method.

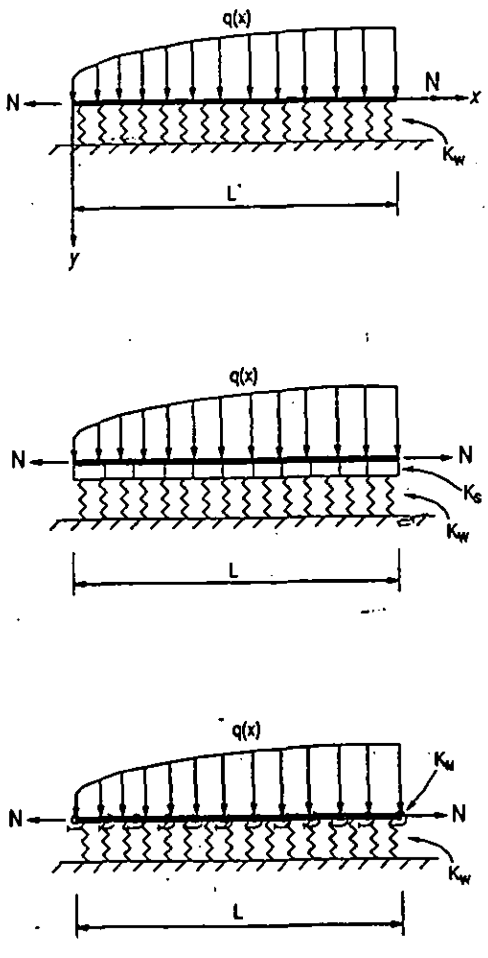

Fig. 1 Beams resting on elastic foundation: (a) Winkler foundation, (b) Pasternak foundation and (c) Generalized foundation

According to the Timoshenko beam theory, the governing equations of free vibration motion of the foregoing beam are given by

$$\kappa GA\left(\frac{d\phi}{dx} - \frac{d^2 w}{dx^2}\right) = q + \sum_{i=1}^{n} P_i \delta(x - e_i)$$
$$+N\frac{d^2 w}{dx^2} + \rho A\omega^2 w - s_1 K_W w$$
$$+s_2 K_S \frac{d^2 w}{dx^2} + s_3 K_V \frac{d^2 w}{dx^2} \quad (2)$$

and

$$EI\frac{d^2\phi}{dx^2} = \kappa GA\left(\phi - \frac{dw}{dx}\right) - \rho I\omega^2\phi$$
$$+s_3 K_M \phi + s_4 K_M \frac{dw}{dx} \quad (3)$$

where $w(x)$ is the transverse displacement, $\phi(x)$ the rotation, κ the shear correction factor, ω the angular frequency, δ the Dirac delta function and the scalar indicators s_i take values as given in Table 1.

For generality and convenience, the following nondimensional terms are introduced:

$$\bar{x} = \frac{x}{L};\ \bar{w} = \frac{w}{L};\ \bar{e} = \frac{e}{L};\ \bar{q} = \frac{qL^3}{EI};$$

$$\bar{P} = \frac{PL^2}{EI};\ \bar{N} = \frac{NL^2}{EI};\ \bar{\omega}^2 = \frac{\omega^2 \rho A L^4}{EI};$$

$$\Psi = \frac{I}{AL^2};\ \Omega = \frac{EI}{\kappa GAL^2};\ \bar{K}_W = \frac{K_W L^4}{EI};$$

$$\bar{K}_S = \frac{K_S L^2}{EI};\ \bar{K}_M = \frac{K_M L^2}{EI};\ \bar{K}_V = \frac{K_V L^2}{EI}. \quad (4)$$

It should be noted that Ω is the shear deformation parameter and for Euler-Bernoulli beams, $\Omega = 0$.

By eliminating the rotation function $\phi(x)$, the governing Eqs. (2) and (3) may be expressed in the form of a fourth order differential equation:

$$\frac{d^4\bar{w}}{d\bar{x}^4} - \lambda\frac{d^2\bar{w}}{d\bar{x}^2} - \alpha\bar{w}$$
$$= \left(\frac{1-\Omega\eta_1}{1+\Omega\eta_2}\right)\left[\bar{q} + \sum_{i=1}^{n}\bar{P}_i\delta(\bar{x} - \bar{e}_i)\right] - \frac{\Omega}{1+\Omega\eta_2}\frac{d^2\bar{q}}{d\bar{x}^2} \quad (5)$$

where

$$\lambda = \frac{\eta_2(1-\Omega\eta_1) - \Omega\eta_3 + \eta_4}{1+\Omega\eta_2};\ \alpha = \frac{\eta_3(1-\Omega\eta_1)}{1+\Omega\eta_2} \quad \text{(6a,b)}$$

$$\eta_1 = \Psi\bar{\omega}^2 - s_3\bar{K}_M;\ \eta_2 = \bar{N} + s_2\bar{K}_S + s_3\bar{K}_V$$

$$\eta_3 = \bar{\omega}^2 - s_1\bar{K}_W;\ \eta_4 = s_4\bar{K}_M - \eta_1 \quad \text{(7a,b,c,d)}$$

The commonly used boundary conditions for Eq. (5), obtained from using Hamiltonian Principle, are summarized in Table 2.

GREEN'S FUNCTIONS FOR BEAMS

The solution to Eq. (5) is given by

$$\overline{w}(\overline{x}) = \left(\frac{1-\Omega\eta_1}{1+\Omega\eta_2}\right)\left[\int_0^1 \mathrm{G}(\overline{x},\xi)\,\overline{q}(\xi)\;d\xi + \sum_{i=1}^{n} \overline{P}_i \mathrm{G}(\overline{x},\overline{e}=\overline{e}_i)\right] \qquad (8)$$

where the Green's function $\mathrm{G}(\overline{x},\xi)$ for the beam must satisfy the boundary conditions given in Table 2 and this equation (Lueschen *et al* 1996)

$$\frac{d^4\mathrm{G}}{d\overline{x}^4} - \lambda\frac{d^2\mathrm{G}}{d\overline{x}^2} - \alpha\mathrm{G} = \delta(\overline{x}-\xi) \qquad (9)$$

Based on the method of initial parameters (Bergman and Hyatt 1989), the general Green's function for the considered beam problem is given by

$$\begin{aligned}\mathrm{G}(\overline{x},\xi) = \frac{1}{a^2+b^2}\{ & W_0(T_1\cosh a\overline{x} + T_2\cos b\overline{x}) \\ & +\theta_0(T_3\sinh a\overline{x} + T_4\sin b\overline{x}) \\ & -M_0(\cos b\overline{x} - \cosh a\overline{x}) \\ & +Q_0(T_5\sinh a\overline{x} + T_6\sin b\overline{x}) \\ & - H(\overline{x}-\xi)[T_5\sinh a(\overline{x}-\xi) + T_6\sin b(\overline{x}-\xi)]\}\end{aligned} \qquad (10)$$

where H is the Heaviside step function, W_0, θ_0, M_0, Q_0 are, respectively, the deflection, rotation, bending moment and transverse shear force at $\overline{x}=0$ and

$$a = \sqrt{\frac{\sqrt{\lambda^2+4\alpha}+\lambda}{2}};\quad b = \sqrt{\frac{\sqrt{\lambda^2+4\alpha}-\lambda}{2}}; \qquad (11a,b)$$

$$T_1 = b^2 - \frac{\Omega\eta_3}{1+\Omega\eta_2};\quad T_2 = a^2 + \frac{\Omega\eta_3}{1+\Omega\eta_2}; \qquad (12a,b)$$

$$T_3 = \frac{1}{a}\left(T_1 + \frac{\eta_4}{1+\Omega\eta_2}\right);\quad T_4 = \frac{1}{b}\left(T_2 - \frac{\eta_4}{1+\Omega\eta_2}\right); \qquad (12c,d)$$

$$T_5 = \frac{1}{a}\left(\frac{1-\Omega\eta_1-\eta_4\Omega}{1+\Omega\eta_2} - \Omega T_1\right);$$

$$T_6 = -\frac{1}{b}\left(\frac{1-\Omega\eta_1-\eta_4\Omega}{1+\Omega\eta_2} + \Omega T_2\right) \qquad (12e,f)$$

Six combinations of end conditions for transversely and axially loaded beams on an elastic foundation are considered. As shown below, the four unknowns $(W_0, \theta_0, M_0, Q_0)$ can be determined from the boundary conditions given in Table 2.

Case 1: Simply supported beams (S-S beams)

$$W_0 = M_0 = 0 \qquad (13)$$

$$\theta_0 = \frac{T_3T_4}{\Delta_{ss}}[\sinh a\sin b(1-\xi) - \sin b\sinh a(1-\xi)] \qquad (14)$$

$$Q_0 = \frac{1}{\Delta_{ss}}[T_4T_5\sin b\sinh a(1-\xi) - T_3T_6\sinh a\sin b(1-\xi)] \qquad (15)$$

where the determinant Δ_{ss} is given by

$$\Delta_{ss} = (T_4T_5 - T_3T_6)\sin b\sinh a \qquad (16)$$

Case 2: Clamped-simply supported beams (C-S beams)

$$W_0 = \theta_0 = 0 \qquad (17)$$

$$M_0 = \frac{T_5T_6}{\Delta_{cs}}[-\sinh a\sin b(1-\xi) + \sin b\sinh a(1-\xi)] \qquad (18)$$

$$Q_0 = \frac{1}{\Delta_{cs}}[T_6\cosh a\sin b(1-\xi) + T_5\cos b\sinh a(1-\xi)] \qquad (19)$$

where the determinant Δ_{cs} is given by

$$\Delta_{cs} = T_6\cosh a\sin b + T_5\cos b\sinh a \qquad (20)$$

Case 3: Clamped beams (C-C beams)

$$W_0 = \theta_0 = 0 \qquad (21)$$

$$\begin{aligned}M_0 = \frac{1}{\Delta_{cc}}\Big\{ & -\frac{T_2T_5^2}{a}\sinh a\xi - \frac{T_1T_6^2}{b}\sin b\xi \\ & -\frac{T_1T_5T_6}{b}[\cos b(1-\xi)\sinh a - \cos b\sinh a(1-\xi)] \\ & +\frac{T_2T_5T_6}{a}[\sin b(1-\xi)\cosh a - \sin b\cosh a(1-\xi)]\Big\}\end{aligned} \qquad (22)$$

$$Q_0 = \frac{1}{\Delta_{cc}}\left\{\left[\frac{T_1T_6}{b}\cos b(1-\xi) + \frac{T_2T_5}{a}\cosh a(1-\xi)\right] \times \left[\cosh a - \cos b\right] - \left[T_6 \sin b(1-\xi) + T_5 \sinh a(1-\xi)\right] \times \left[\frac{T_1}{b}\sin b + \frac{T_2}{a}\sinh a\right]\right\} \tag{23}$$

where the determinant Δ_{cc} is given by

$$\Delta_{cc} = \left(\frac{T_2T_5}{a} - \frac{T_1T_6}{b}\right)(1-\cos b\cosh a) - \left(\frac{T_2T_6}{a} + \frac{T_1T_5}{b}\right)\sin b \sinh a \tag{24}$$

Case 4: Clamped-free beams (C-F beams)

$$W_0 = \theta_0 = 0 \tag{25}$$

$$M_0 = \frac{1}{\Delta_{cf}}\left\{-\frac{T_1T_6^2}{b}\sin b\xi - \frac{T_2T_5^2}{a}\sinh a\xi + \frac{T_1T_5T_6}{a}\left[\sin b\cosh a(1-\xi) - \sin b(1-\xi)\cosh a\right] + \frac{T_2T_5T_6}{b}\left[\cos b(1-\xi)\sinh a - \cos b \sinh a(1-\xi)\right]\right\} \tag{26}$$

$$Q_0 = \frac{1}{\Delta_{cf}}\left\{-\frac{T_1T_6}{b}\cos b\xi + \frac{T_2T_5}{a}\cosh a\xi - \frac{T_2}{b}\left[T_6\cos b(1-\xi)\cosh a - T_5 \sinh a(1-\xi)\sin b\right] + \frac{T_1}{a}\left[T_5\cosh a(1-\xi)\cos b + T_4 \sin b(1-\xi)\sinh a\right]\right\} \tag{27}$$

where the determinant Δ_{cf} is given by

$$\Delta_{cf} = \left(\frac{T_1T_5}{a} - \frac{T_2T_6}{b}\right)\cos b\cosh a + \left(\frac{T_2T_5}{b} + \frac{T_1T_6}{a}\right)\sin b\sinh a - \frac{T_1T_6}{b} + \frac{T_2T_5}{a} \tag{28}$$

Case 5: Simply supported-free beams (S-F beams)

$$W_0 = M_0 = 0 \tag{29}$$

$$\theta_0 = \frac{1}{\Delta_{sf}}\left\{-\frac{T_1T_6^2}{b}\sin b\xi - \frac{T_2T_5^2}{a}\sinh a\xi - \frac{T_1T_5T_6}{a}\left[\sin b(1-\xi)\cosh a - \sin b\cosh a(1-\xi)\right] + \frac{T_2T_5T_6}{b}\left[\cos b(1-\xi)\sinh a - \cos b\sinh a(1-\xi)\right]\right\} \tag{30}$$

$$Q_0 = \frac{1}{\Delta_{sf}}\left\{\frac{T_1T_4T_6}{b}\sin b\xi + \frac{T_2T_3T_5}{a}\sinh a\xi + \frac{T_2}{b}\left[T_4T_6\cos b\sinh a(1-\xi) - T_3T_6\sinh a\cos b(1-\xi)\right] + \frac{T_1}{a}\left[T_3T_6\sin b(1-\xi)\cosh a - T_4T_5\cosh a(1-\xi)\sin b\right]\right\} \tag{31}$$

where the determinant Δ_{sf} is given by

$$\Delta_{sf} = (T_3T_6 - T_4T_5)\left(\frac{T_1}{a}\cosh a\sin b - \frac{T_2}{b}\sinh a\cos b\right) \tag{32}$$

Case 6: Free-free beams (F-F beams)

$$M_0 = Q_0 = 0 \tag{33}$$

$$W_0 = \frac{1}{\Delta_{ff}}\left\{-\frac{T_1T_4T_6}{b}\sin b\xi - \frac{T_2T_3T_5}{a}\sinh a\xi + T_4T_5\left[\frac{T_1}{a}\sin b\cosh a(1-\xi) - \frac{T_2}{b}\cos b\sinh a(1-\xi)\right] + T_3T_6\left[\frac{T_2}{b}\cos b(1-\xi)\sinh a - \frac{T_1}{a}\sin b(1-\xi)\cosh a\right]\right\} \tag{34}$$

$$\theta_0 = \frac{1}{\Delta_{ff}}\left\{\frac{T_1T_2T_6}{b}\cos b\xi + \frac{T_1T_2T_5}{a}\cosh a\xi + T_1T_6\left[\frac{T_1}{a}\sin b(1-\xi)\sinh a - \frac{T_2}{b}\cos b(1-\xi)\cosh a\right] + T_2T_5\left[\frac{T_2}{b}\sinh a(1-\xi)\sin b + \frac{T_1}{a}\cosh a(1-\xi)\cos b\right]\right\} \tag{35}$$

where the determinant Δ_{ff} is given by

$$\Delta_{ff} = \left(\frac{T_1T_2T_4}{b} + \frac{T_1T_2T_3}{a}\right)(1-\cos b\cosh a) + \left(\frac{T_1^2T_4}{a} - \frac{T_2^2T_3}{b}\right)\sin b\sinh a \tag{36}$$

BENDING PROBLEM

For the bending problem of the Timoshenko beam under transverse load and the axial load, the angular frequency is set to zero, i.e. the parameters η_1, η_3, η_4 reduce to

$$\eta_1 = -s_3\overline{K}_M;\ \ \eta_3 = -s_1\overline{K}_w;\ \ \eta_4 = (s_3 + s_4)\overline{K}_M \qquad \text{(37a,b,c)}$$

The transverse deflection is determined from Eqs. (8) and (10) and depending on the end conditions, the appropriate expressions given in Eqs. (13)-(36) for

W_0, θ_0, M_0, Q_0. The stress-resultants can be determined from the following relations:

$$
\begin{aligned}
\overline{M}(\overline{x}) &= \frac{M(x)L}{EI} = \frac{d\phi}{d\overline{x}} \\
&= \Omega\left(\overline{q} + \sum_{i=1}^{n} \overline{P}_i \delta(\overline{x} - \overline{e}_i)\right) + \Omega\eta_3 \overline{w} + (1 + \Omega\eta_2)\frac{d^2\overline{w}}{d\overline{x}^2} \\
&= \Omega\left(\overline{q} + \sum_{i=1}^{n} \overline{P}_i \delta(\overline{x} - \overline{e}_i)\right) \\
&+ \Omega\eta_3\left[\left(\frac{1-\Omega\eta_1}{1+\Omega\eta_2}\right)\left(\int_0^1 \mathrm{G}(\overline{x},\xi)\overline{q}(\xi)\, d\xi\right.\right. \\
&\left.\left.+ \sum_{i=1}^{n} \overline{P}_i \mathrm{G}(\overline{x}, \overline{e} = \overline{e}_i)\right)\right] + (1 - \Omega\eta_1)\left[\frac{\partial^2}{\partial\overline{x}^2}\int_0^1 \{\mathrm{G}(\overline{x},\xi)\}\overline{q}(\xi) d\xi\right. \\
&\left.+ \sum_{i=1}^{n} \overline{P}_i \frac{\partial^2}{\partial\overline{x}^2}\mathrm{G}(\overline{x}, \overline{e} = \overline{e}_i)\right]
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
\overline{Q}(\overline{x}) &= \frac{Q(x)L^2}{EI} = \left(\frac{1+\Omega\eta_2}{1-\Omega\eta_1}\right)\frac{d^3\overline{w}}{d\overline{x}^3} \\
&+ \left(\frac{\Omega\eta_3 - \eta_4}{1-\Omega\eta_1}\right)\frac{d\overline{w}}{d\overline{x}} + \left(\frac{\Omega}{1-\Omega\eta_1}\right)\frac{d\overline{q}}{d\overline{x}} \\
&= \frac{\partial^3}{\partial\overline{x}^3}\int_0^1 \{\mathrm{G}(\overline{x},\xi)\overline{q}(\xi)\} d\xi \\
&+ \sum_{i=1}^{n} \overline{P}_i \frac{\partial^3}{\partial\overline{x}^3}\mathrm{G}(\overline{x}, \overline{e}_i) + \left(\frac{\Omega}{1-\Omega\eta_1}\right)\frac{d\overline{q}}{d\overline{x}} \\
&+ \left(\frac{\Omega\eta_3 - \eta_4}{1+\Omega\eta_3}\right)\left[\frac{\partial}{\partial\overline{x}}\int_0^1 \{\mathrm{G}(\overline{x},\xi)\}\overline{q}(\xi) d\xi\right. \\
&\left.+ \sum_{i=1}^{n} \overline{P}_i \frac{\partial}{\partial\overline{x}}\mathrm{G}(\overline{x}, \overline{e}_i)\right]
\end{aligned}
\tag{39}
$$

VIBRATION AND BUCKLING PROBLEMS

In the cases of vibration and buckling problems, the exact vibration and stability criteria of Timoshenko beams for the considered six combinations of end conditions are obtained by setting the determinant Δ to zero.

Note that for buckling problem, the frequency parameter $\overline{\omega}^3$ is set to zero and the axial tensile preload $\overline{N}$ is changed to $\overline{N}_c = -\overline{N}$ so as to denote a compressive axial load. The parameters η_1, η_3, η_4 reduce to those given in Eqs. (35a)-(35c), respectively.

CONCLUDING REMARKS

Using Green's functions, exact solutions have been derived for the bending, buckling and vibration problems of Timoshenko beams on various elastic foundation models. All the combinations of classical end conditions for beams were considered. The classical Euler-Bernoulli beam solutions may be obtained from these exact solutions by setting the shear deformation parameter $\Omega = 0$ and neglecting the effect of rotary inertia. The exact solutions should be useful in providing benchmark results for checking the validity, convergence and accuracy of numerical results.

It is interesting to note that the exact solutions for the bending of Timoshenko beams on Winkler foundation may be applied to thick cylindrical shells under axisymmetric loading. The analogy between these two problems was pointed out by Ma and Pulmano (1996).

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Table 1 Scalar indicator s_i

Foundation Type	s_1	s_2	s_3	s_4	s_5
Winkler	1	0	0	0	0
Pasternak	1	1	0	0	0
Generalized (Type 1)	1	0	1	0	0
Generalized (Type 2)	1	0	0	1	0
Vlasov	1	0	0	0	1
No elastic foundation	0	0	0	0	0

Table 2 Boundary conditions for beams

Types of end condition	Boundary Conditions
Simple (S)	$\overline{w}=0,\ \frac{d^2\overline{w}}{d\overline{x}^2}=0$
Clamped (C)	$\overline{w}=0,\quad \Omega(1+\eta_2\Omega)\frac{d^3\overline{w}}{d\overline{x}^3}+(1-\Omega\eta_1+\eta_3\Omega^2-\eta_4\Omega)\frac{d\overline{w}}{d\overline{x}}=0$
Free (F)	$(1+\Omega\eta_2)\frac{d^2\overline{w}}{d\overline{x}^2}+\Omega\eta_3\overline{w}=0,$ $(1+\eta_2\Omega)\frac{d^3\overline{w}}{d\overline{x}^3}+[\Omega\eta_3-\eta_4-(1-\Omega\eta_1)\eta_2]\frac{d\overline{w}}{d\overline{x}}=0$