THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
345 E. 47th St., New York, N.Y. 10017

97-AA-104

BUCKLING OF CIRCULAR PLATES BASED ON REDDY PLATE THEORY

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ABSTRACT: Treated herein is the elastic buckling of circular plates based on the Reddy plate theory. This plate theroy extends the Kirchhoff (or the classical thin) plate theory to allow for the effect of transverse shear deformation. Unlike the Mindlin's shear deformation plate theory, there is no need for a shear correction factor in the Reddy plate theory. In this paper, exact buckling solutions are derived for circular plates whose edges are simply supported and elastically restrained against rotation as well. This general edge condition includes the classical simply supported and clamped edges at the limiting values of the elastic rotational restraint constant. The buckling solutions are expressed in terms of the well-known Kichhoff buckling solutions. A comparison of buckling loads between the Mindlin, Reddy and three-dimensional elasticity plates is also given.

INTRODUCTION

When dealing with thick plates, the application of the Kirchhoff (or classical thin) plate theory leads to an underprediction of the deflections and an overprediction of the buckling loads and natural frequencies. This is because the effect of transverse shear deformation becomes significant in such thick plates and thus cannot be ignored. Reissner (1945) and Mindlin (1951) proposed a simple plate theory to allow for this effect by relaxing the normality assumption so that the normal to the midsurface before deformation need not be perpendicular to the plate midsurface after deformation. The straightness of the normal is, however, still maintained. The Reissner-Mindlin plate theory requires the introduction of a shear correction factor to compensate for the error due to the above assumption which results in a constant shear strain (and thus shear stress) through the plate thickness.

The more refined Reddy plate theory (Reddy 1984, 1997) does away with the need for this shear correction factor because the assumed third-order displacement field satisfies the zero shear stress condition at the free surfaces. Adopting the Reddy plate theory for the buckling problem of circular plates, this paper presents the exact buckling loads for uniformly, inplane loaded, circular plates with any homogeneous edge conditions such as simply supported edges, clamped edges, and simply supported edges with elastic rotational restraints. The buckling solutions have been derived using an analogy approach.

Presented at the ASME ASIA '97 Congress & Exhibition
Singapore – September 30–October 2, 1997

STRESS RESULTANT-DISPLACEMENT RELATIONS

Consider an elastic, isotropic circular plate of radius R, uniform thickness h, modulus of elasticity E, shear modulus of rigidity G and Poisson's ratio ν subjected to a uniform radial load N. According to the Reddy plate theory (Reddy 1984, 1997), the stress-resultant-displacement relations for axisymmetric buckling are given by:

$$M_r^R = \frac{4D}{5}\left(\frac{d\phi_r}{dr}+\frac{\nu}{r}\phi_r\right)-\frac{D}{5}\left(\frac{d^2w^R}{dr^2}+\frac{\nu}{r}\frac{dw^R}{dr}\right) \qquad (1)$$

$$M_\theta^R = \frac{4D}{5}\left(\nu\frac{d\phi_r}{dr}+\frac{1}{r}\phi_r\right)-\frac{D}{5}\left(\nu\frac{d^2w^R}{dr^2}+\frac{1}{r}\frac{dw^R}{dr}\right) \qquad (2)$$

$$P_r^R = \frac{4h^2D}{35}\left(\frac{d\phi_r}{dr}+\frac{\nu}{r}\phi_r\right)-\frac{h^2D}{28}\left(\frac{d^2w^R}{dr^2}+\frac{\nu}{r}\frac{dw^R}{dr}\right) \qquad (3)$$

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$$Q_r^R = \frac{2hG}{3}\left(\phi_r+\frac{dw^R}{dr}\right) \qquad (5)$$

$$R_r^R = \frac{h^3G}{30}\left(\phi_r+\frac{dw^R}{dr}\right) \qquad (6)$$

where r is the radial coordinate; w the transverse displacement of the plate midplane; ϕ_r the rotation of the normal; $D = Eh^3/\left[12\left(1-\nu^2\right)\right]$ the flexural rigidity of the plate; M_r, M_θ and P_r, P_θ are, respectively, the moments and the higher-order moments; Q_r and R_r are, respectively, the shear force and the higher-order shear force; and the superscript R denotes the quantities belonging to Reddy plate theory.

EQUILIBRIUM EQUATIONS FOR AXISYMMETRIC BUCKLING

Reddy (1997) derived the general governing equations for the bending, buckling and vibration according to his third order shear deformation theory. The equations, when cast in polar coordinates and specialized for axisymmetric buckling are furnished by

$$\frac{d}{dr}\left(rM_r^R-\frac{4}{3h^2}rP_r^R\right)-\left(M_\theta^R-\frac{4}{3h^2}P_\theta^R\right) = r\left(Q_r^R-\frac{4}{h^2}R_r^R\right) \qquad (7)$$

and

$$\frac{d}{dr}\left(rQ_r^R-r\frac{4}{h^2}R_r^R\right)+\frac{4}{3h^2}r\nabla^2\mathrm{P}^R = N^R r\nabla^2 w^R \qquad (8)$$

where the Laplacian operator $\nabla^2(\bullet)=d^2(\bullet)/dr^2+(1/r)d(\bullet)/dr$ and the higher order moment sum P^R is defined as

$$\mathrm{P}^R = \frac{P_r^R+P_\theta^R}{1+\nu} = \frac{4h^2D}{35}\frac{1}{r}\frac{d}{dr}(r\phi_r)-\frac{h^2D}{28}\nabla^2 w^R \qquad (9)$$

The substitution of Eq. (7) into Eq. (8) leads to

$$\nabla^2\mathrm{M}^R = N^R\nabla^2 w^R \qquad (10)$$

where the moment sum M^R is defined as

$$\mathrm{M}^R = \frac{M_r^R+M_\theta^R}{1+\nu} = \frac{4D}{5}\frac{1}{r}\frac{d}{dr}(r\phi_r)-\frac{D}{5}\nabla^2 w^R \qquad (11)$$

The substitution of Eqs. (5), (6) and (9) into Eq. (8) yields

$$\frac{8Gh}{15}\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)+\nabla^2 w^R\right]+\frac{16D}{105}\nabla^2\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)\right] -\frac{D}{21}\nabla^4 w^R = N^R\nabla^2 w^R \qquad (12)$$

From Eq. (11), we have the relation

$$\nabla^2\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)\right] = \frac{5}{4D}\nabla^2\mathrm{M}^R+\frac{1}{4}\nabla^4 w^R \qquad (13)$$

In view of Eqs. (10), (11) and (13), one may express Eq. (12) as

$$\nabla^4 M^R - \left(\frac{420(1-\nu)}{h^2} - \frac{85}{D}N^R\right)\nabla^2 M^R - \left(\frac{420(1-\nu)}{h^2}N^R\right)M^R = 0 \tag{14}$$

Eq. (14) can be factored to give

$$\left(\nabla^2 + \lambda_1^R\right)\left(\nabla^2 + \lambda_2^R\right)M^R = 0 \tag{15}$$

where

$$\lambda_{1,2}^R = -\left(\frac{210(1-\nu)}{h^2} - \frac{85}{2D}N^R\right) \pm \sqrt{\left(\frac{210(1-\nu)}{h^2} - \frac{85}{2D}N^R\right)^2 + \left(\frac{420(1-\nu)}{h^2}N^R\right)} \tag{16}$$

BOUNDARY CONDITIONS

For axisymmetric buckling of circular plates with homogeneous edge condition, the boundary conditions are

$$M^R = C_1, \quad \text{at } r = R \tag{17}$$

where C_1 is a constant and

$$\frac{dM^R}{dr} = 0 \quad \text{at } r = 0 \tag{18}$$

BUCKLING LOAD RELATIONSHIP

To derive the exact axisymmetric buckling solution of the Reddy circular plate, an analogy method is adopted. The strategy here in to transform the governing equation given by (15) to a form of the well-known Kirchhoff plate buckling equation. First, we let

$$\mu = \left(\nabla^2 + \lambda_i^R\right)M^R \tag{19}$$

where $i = 1$ or 2. In view of Eq. (19), the governing equation (15) may be written as

$$\left(\nabla^2 + \lambda_j^R\right)\mu = 0 \tag{20}$$

where $j = 1 \text{ or } 2$, and $j \neq i$. Noting Eq. (17) and the fact that the moment sum M^R is a function of r, we conclude that

$$\mu = C_2 \quad \text{at } r = R \tag{21}$$

where C_2 is a constant. Also using Eqs. (10) and (18), and the fact that $d\left(\nabla^2 w^R\right)/dr = 0$ at $r = 0$, we can show that

$$\frac{d\mu}{dr} = 0 \quad \text{at } r = 0 \tag{22}$$

Now, in the case of axisymmetric buckling of Kirchhoff circular plates with homogeneous edge condition, Wang (1995, 1996) has shown that the governing equation and the boundary conditions are, respectively

$$\left(\nabla^2 + \lambda^K\right)M^K = 0 \tag{23}$$

and

$$M^K = C_3 \quad \text{at } r = R \tag{24}$$

$$\frac{dM^K}{dr} = 0 \quad \text{at } r = 0 \tag{25}$$

where

$$\lambda^K = \frac{N^K}{D} \tag{26}$$

and

$$M^K = \frac{M_r^K + M_\theta^K}{1+\nu} = -D\left(\nabla^2 w^K\right) \tag{27}$$

The superscript K in Eqs. (23) to (27) denotes quantities belonging to the Kirchhoff plate.

Comparing the similar form of the governing equations and boundary conditions of Reddy and

Kirchhoff plates given respectively by Eqs. (20)-(22) and by Eqs. (23)-(25), it can be deduced that

$$\lambda_j^R = \lambda^K \tag{28}$$

The substitution of Eqs. (16) and (26) into Eq. (28) furnishes the buckling load relationship given by

$$N^R = \frac{N^K\left(1+\dfrac{N^K}{70Gh}\right)}{1+\dfrac{N^K}{\frac{14}{17}Gh}} \tag{29}$$

where only λ_1^R is used because the negative value of λ_2^R does not lead to a feasible buckling solution. The foregoing relationship given in Eq. (29) is valid for circular plates with any homogeneous edge condition such as simply supported edges, clamped edges, and simply supported edges with elastic rotational restraints. With this buckling load relationship, the Reddy buckling loads may be readily obtained as the exact Kirchhofff buckling loads for these circular plates are already derived previously by Reismann (1952) and Kerr (1962). Now, the Kirchhoff buckling solution for these plate cases may be unifiedly expressed as

$$\sqrt{\frac{N^K R^2}{D}} J_0\left(\sqrt{\frac{N^K R^2}{D}}\right) + \left[\frac{K_r R}{D} - (1-\nu)\right] J_1\left(\sqrt{\frac{N^K R^2}{D}}\right) = 0 \tag{30}$$

where $J_0(\bullet)$ and $J_1(\bullet)$ are Bessel functions of the first kind of order 0 and 1, respectively, and K_r is the rotational spring stiffness with extreme values covering the two ideal edges of simply supported $(K_r = 0)$ and clamped $(K_r = \infty)$.

COMPARISON OF BUCKLING RESULTS FROM DIFFERENT PLATE THEORIES

When one adopts the Mindlin plate theory, the relationship takes the form of (Wang 1995, 1996)

$$N^M = \frac{N^K}{1+\dfrac{N^K}{\kappa^2 Gh}} \tag{31}$$

where the N^M denotes the Mindlin buckling load and κ^2 is the shear correction factor.

Recently, Ye (1995) derived the buckling load of circular plates from three-dimensional elasticity considerations. The analysis is based on a recursive formulation that results in the need to solve for only the roots of a 2 X 2 determinant for the buckling load.

Table 1 presents the Kirchhoff, Mindlin, Reddy and Ye's buckling load factors NR^2/D for circular plates with various values of the thickness to radius ratio h/R, elastic rotational restraint parameter $K_r R/D$ and Poisson's ratio $\nu = 0.3$. Note that the Kirchhoff buckling load factor is independent of h/R due to the neglect of transverse shear deformation. A shear correction factor $\kappa^2 = 5/6$ has been assumed for the Mindlin plates. Both the Mindlin and Reddy results are very close to each other but are somewhat lower than the three-dimensional elasticity solutions of Ye (1995). The Reddy plate theory has the advantage over the Mindlin plate theory in that there is no need for a shear correction factor.

CONCLUDING REMARKS

The exact Reddy buckling loads for axisymmetric circular plates have been derived. The plate edges may be simply supported, clamped or simply supported with elastic rotational restraints. The solutions are expressed in an exact canonical form involving the well-known exact Kirchhoff plate buckling solutions. A comparison of Reddy and

Mindlin buckling load factors show that they are practically the same when the Mindlin shear correction factor κ^2 is assumed to be $5/6$. Both plate theories give lower buckling factors when compared to the three-dimensional elasticity results.

The advantage of the exact relationship derived herein is that it can be readily modified for symmetric laminated circular plates with isotropic layers by using the appropriate stiffenesses. While this can also be performed for the Mindlin plate theory, it requires the specification of appropriate shear correction factors which are not readily available.

ACKNOWLEDGMENT

The project is funded by research Grant RP 950684 made available by The National University of Singapore.

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Table 1 Comparison of buckling load factors for circular plates based on different theories

$\frac{h}{R}$	$\frac{K_r R}{D}$	Kirchhoff Plate Theory	Mindlin Plate Theory	Reddy Plate Theory	Ye's 3-D Elasticity Results
0.05	0	4.1978	4.1853	4.1853	[illegible]
	1	6.3532	6.3245	6.3245	[illegible]
	10	12.173	12.068	12.068	[illegible]
	∞	14.682	14.530	14.530	14.552
0.10	0	4.1978	4.1481	4.1481	[illegible]
	1	6.3532	6.2399	6.2400	[illegible]
	10	12.173	11.764	11.764	[illegible]
	∞	14.682	14.091	14.091	14.177
0.20	0	4.1978	4.0056	4.0057	[illegible]
	1	6.3532	5.9231	5.9235	[illegible]
	10	12.173	10.686	10.688	[illegible]
	∞	14.682	12.572	12.576	12.824
0.30	0	4.1978	3.7888	3.7893	[illegible]
	1	6.3532	5.4610	5.4625	[illegible]
	10	12.173	9.2710	9.2792	[illegible]
	∞	14.682	10.658	10.671	11.024