VIBRATION OF A ROTOR WITH EXTREMELY NONLINEAR STIFFNESS PASSING THROUGH CRITICAL SPEEDS

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ABSTRACT
Soft spring nonlinearity refers to supporting stiffness becomes smaller with increase of vibratory displacements. In this paper, transient response and subharmonic vibration of a rotor with extremely soft nonlinear spring characteristics are investigated. A simple but exact computer model of the phenomena has been evolved based on the numerical integration of a finite difference formulation. Response curves and wave forms of the rotor vibration passing through critical speeds are computed. The subharmonic vibration of the rotor with soft nonlinear spring characteristics has been observed by spectral analysis of vibratory response. It is shown that the rotor will tend to bounce when its vibratory displacements exceed the offset distance of the system with stiffness $K_1$ when it is accelerated for passing through critical speeds and the vibratory motion after bouncing is predominantly at the forcing frequency. Maximum amplitude of transient response is affected mainly by damping ratio $\xi$, the degree of nonlinearity $\beta$ and the offset distance $h$. A series of subharmonic pseudo critical peaks will appear in the acceleration of a rotor with soft nonlinear spring characteristics. But the general nature of the subharmonic vibration is much different from that with hard ones. Finally some computational results are demonstrated by an experiment.

NOMENCLATURE

- $A,a$: response amplitude
- $C,c$: damping coefficient
- $E,e$: unbalance eccentricity
- $G,g,g'$: spring force component
- $H,h$: offset distance of the linear system with stiffness $K_1$
- $K_1,k_1$: spring stiffness within the offset distance
- $K_2,k_2$: total spring stiffness beyond the offset distance
- $M,m$: rotor point mass
- $n$: order number of subharmonic vibration or integer
- $T,t$: time
- $X,x$: displacement in $x$ direction
- $Y,y$: displacement in $y$ direction
- $\alpha$: acceleration
- $\beta$: the degree of nonlinearity $K_1/K_2$
- $\epsilon$: little parameter
- $\gamma$: frequency ratio $\Omega/\Omega_1$
- $\lambda$: acceleration ratio $\alpha/\Omega_1^2$
- $\Omega,\omega$: frequency
- $\xi$: damping ratio

INTRODUCTION
Modern complex rotating machinery contains various sources of strong nonlinearities, which include clearances in bearings, gears and splines, rubbing in splines, seals and rotor blades, some dampers (such as squeeze oil film and other nonlinear dampers) and fluid effects. Previous investigations focus on the characteristics of a rotor with hard spring nonlinearity. Observed hard nonlinear behavior of actual rotor systems includes jump discontinuities (Ehrich 1966), large subsynchronous motion (Bently 1979, Ishida, 1988), quasi-periodic and possible chaos (Ehrich, 1991). Nataraj and Nelson (1987) have made a great progress in solution method for determining the periodic response of larger, multi disk rotor systems. They utilized a subsystem approach to reduce the size of the resulting system of algebraic equations.

With the development of industry, more and more rotor...
systems with soft nonlinear spring characteristics (especially, the rotor systems with soft nonlinear stiffness dampers) are being used. Figure 1 shows the spring curves of steel wire dampers developed by Hu (1988) and used in attenuation of vibration of a shaft in ship, which has obviously soft spring nonlinearity. Authors have developed a support with soft spring nonlinearity for suppressing vibratory response when a rotor is accelerated for passing through critical speeds (1992). Some supporting bearing clearances as shown in Fig.2, which were noted by Ehrich (1966) and measured in an engine with a dual-rotor, have also soft spring nonlinearity. But few published papers discussed or analyzed the general nature of transient response and subharmonic vibration of a rotor system with soft nonlinear spring characteristics, which is much different from that of a hard nonlinear spring system. This encourages us to take a more intensive looking at the transient response and subharmonic vibration of a soft nonlinear spring system.

**TRANSIENT RESPONSE**

The rotor with soft nonlinear spring characteristics is modeled numerically as a single mass, two degree-of-freedom system with a damping ratio

\[ \xi_1 = \frac{C}{2(K_1 M)^{\frac{1}{2}}} \]

The system nonlinearity are measured by the ratio of stiffnesses

\[ \beta = \frac{K_1}{K_2} \]

\( \beta > 1, \beta = 1, \beta < 1 \) indicate a soft nonlinear system, a linear system and a hard nonlinear system respectively.

The steady state periodic response of the rotor system is determined as summarized in the Appendices. Figure 3 shows the computed steady state response for different \( \beta \). The response peaks deflect towards right for a hard nonlinear system and towards left for a soft nonlinear system. In general, the maximum response amplitude of a soft nonlinear system is smaller than that of the linear system with stiffness \( k_1 \). Figure 4(a) and 4(b) show typical transient response, its spectrum and trajectory of a rotor system with soft nonlinear spring characteristics when it is accelerated for passing through critical speeds. The dash line in Fig.4(a) indicates steady state response of the same rotor system. It is found that when a rotor system is accelerated for passing through critical speeds in small acceleration and \( \omega_s \) is more less than \( \omega_d \), no matter how great initial rotational speed \( \omega_d \) and acceleration \( \alpha \) are, the vibratory response always passes along \( L_1 \) line and bounces near \( \omega_d \) (refer to Fig.3). The vibratory amplitude after bouncing is greater than that determined by the steady state response and oscillates near it.
the smaller $\beta$ and the greater $\zeta$, the oscillation will attenuate and stabilize at the steady state response. The phenomena can be explained as follow. The vibration after bouncing is transition from linear stationary vibration to nonlinear one. The transient vibration consists of synchronous vibration at forcing frequency ratio $0.72$ and asynchronous vibration at natural frequency ratio $0.45$. Dominant part of the transient vibration is synchronous (refer to Fig.4(b)). Due to structural damping, the asynchronous response attenuates gradually with increase of time. Then, effect of acceleration on vibratory response near critical speeds is greater than that far from critical speeds. Generally, the rotational speed when vibratory response bounces is far from both the critical speeds in the linear system with smaller stiffness $(k_2)$ and those with larger one $(k_1)$. So, effect of acceleration can’t be took into account when $a$ is small and the vibratory response always passes along stationary curve.

The maximum amplitude of transient response of a rotor system with soft nonlinear spring characteristics is determined by acceleration $a$, the degree of nonlinearity $\beta$, the offset distance of the linear system $h$ etc. when it is accelerated for passing through critical speeds. Their effects on maximum amplitude are shown in Fig.5(a) and 5(b). Figure 5(a) shows the influence of acceleration $a$ for different $\beta$. With variation of acceleration $a$, the maximum amplitude is kept nearly a constant. This characteristic is very different from those of a linear rotor system and a hard nonlinear rotor system. In those systems, the maximum amplitude decreases rapidly with increase of acceleration. With increase of the degree of nonlinearity $\beta$, the maximum amplitude decreases gradually. Figure 5(b) shows the influence of offset distance of the linear system $h$. With increase of $h$, there is a minimum value of maximum amplitude at $h_0$. With increase and decrease of $h$ from $h_0$, all maximum amplitudes of transient response will rise. For example, the $h_0$ of the system, which the degree of nonlinearity $\beta$ is equal to 4.17, is 0.85. If $h$ is reduced to 0.5, the maximum amplitude will be 5.9 and if $h$ is increased to 4.0, the maximum amplitude will rise to 9.2. The value $h_0$ is very important for designing bearing supports with nonlinear spring characteristics.

The vibrational circumstances of a rotor system will be more complex when the system is decelerated. The beginning rotational speed of nonlinear vibration is defined as $\omega_b$ and the ending speed is defined as $\omega_f$. Within the nonlinear response zone $\omega_b > \omega > \omega_f$, there may be 1, 2 or 3 vibrational levels. The level reached by a rotor vibration is dependent on the initial motion conditions that include rotational acceleration, rotational speed and a period of standing time. If an initial speed $\omega_0 > \omega_b$, the vibratory response is always along L2 line and jumps to the lower vibrational level at $\omega_f$ (refer to Fig.3 or Fig.6). The vibratory amplitude after jumping will oscillate near the steady state response for a period of time and then stabilize at it for small acceleration. The period of time is determined by damping $\zeta$, the degree of nonlinearity $\beta$ and the offset distance $h$. The greater the deceleration, the smaller the maximum transient response is and the later the vibratory response jumps (as show in Fig.6). The vibratory response is mainly synchronous. If $\omega_b > \omega_f$ and $\omega_0 > \omega_f$, the transient response may jump to the lower level in many circumstances at the beginning to decelerate. Sometimes it may be kept at the higher level and then jumps to the lower level near $\omega_f$ for small deceleration. If the rotor is stopped being decelerated at $\omega_f < \omega_0 < \omega_b$, the transient response may jump to the lower level in many circumstances and if the rotor continues to be decelerated, it may maintain in the lower level. Why the response differs depending on whether $\omega_0 < \omega_b$ and
$\omega_0 > \omega_2$. The phenomena result from vibratory energy. Although the rotor can vibrate stably in both higher and lower levels, the energy of a rotor vibrating in higher level is greater than that in lower one and vibration in lower level is more stable. If a rotor suffers a disturbance in $\omega_r < \omega_0 < \omega_2$ such as stopping or starting decelerating, the vibratory response will jump to a more stable level.

SUBHARMONIC VIBRATION

Subharmonic vibration refers to the response of a dynamic system to excitation at a whole-number-multiple (a) of its natural frequency by vibration asynchronously at its natural frequency, that is, at $(1/a)$ of the excitation. The phenomenon was first observed in hard nonlinear spring rotor systems. The highest order subharmonic vibration ($1/8$, $1/9$) has been investigated by Ehrich (1988) experimentally and theoretically. But no papers involved the subharmonic vibration of a soft nonlinear rotor system, whose characteristics are very different from those with hard ones.

Figure 7(a) shows a typical amplitude and frequency response of subharmonic vibration. From Fig.7 (a), we can find two additional pseudo-criticals at uniform intervals at two times and three times the fundamental critical speed. The whole frequency range can be divided into several zones: synchronously vibrational zones, a $1/2$ subharmonic vibrational zone and a $1/3$ subharmonic vibrational zone. The dominant response frequencies are forcing frequency, $1/2$ forcing frequency and $1/3$ forcing frequency respectively. For the greater $\beta$ and greater damping $\xi$, there is an obviously synchronously vibrational zone between $1/2$ zone and $1/3$ zone. The phenomenon has never been observed in a hard nonlinear spring system. The results of spectral analysis and trajectories in transition between predominantly 2nd and predominantly 3rd subharmonic vibration in Fig.7(b) confirm the result. As shown in Fig.7(b), when the rotational speed ratio is higher than 1.17, the 2nd order subharmonic vibration disappears rapidly and the trajectory becomes a circle. The rotor vibrates at forcing frequency. When $\gamma > 1.65$, the $1/3$ subharmonic vibration will be main component. For high nonlinearity $\beta$ and low damping $\xi$, the transitional behavior is more complex as the rotor motion changes from predominantly $1/n$ per revolution to $1/(n+1)$ per revolution. The phenomenon has never been observed in a hard nonlinear spring system. In a hard nonlinear spring system, there is chaotic vibration in the transition zone midway between adjacent zones of subharmonic response and main component is synchronous vibration at natural frequency. But the main component is not at natural frequency or at forcing frequency in soft spring system. The vibratory frequency and amplitude are determined by disturbance besides motion parameters such as acceleration and rotational speed. The vibration may be very dangerous for a real rotor in an operating machinery.

Our analysis also provide an explanation of effects which
was absent in the literature of subharmonic response in a hard nonlinear spring rotor system. For a soft nonlinear system, the occurrence and amplitude of subharmonic vibration is affected not only by damping ratio \( \xi \) and nonlinearity \( \beta \), but also by acceleration \( a \) and the offset distance of the linear system \( h \). For small acceleration \( a \), some orders of subharmonic vibration will disappear. Figure 9 shows the effect of increasing acceleration. The \( 1/2 \) subharmonic vibration would not be encountered when the acceleration ratio \( \lambda \) is smaller than 2.41x10\(^{-4}\). Figure 10 shows the effect of damping \( \xi \) on the subharmonic pseudo-critical peak amplitude for moderately soft nonlinear systems. With increase of damping, the subharmonic response decreases greatly and some order subharmonic vibration disappears.

But if the degree of nonlinearity is very high, the increase of damping \( \xi \) will not result in the disappearance of subharmonic vibration (Fig.10(b) \( \beta = 10 \)).

In the literature published previously, there was few mention in the effect of offset distance of the linear system on subharmonic vibration. Through our investigation, it is shown that offset distance of the linear system \( h \) affects the occurrence and amplitude of subharmonic vibration. Figure 11(a) shows the effect of offset distance of the linear system \( h \) on subharmonic response of three moderately nonlinear systems. For a more linear system \( \beta = 3.33 \), the third order subharmonic vibration occurs only when \( h > 0.75 \). For more nonlinear system \( \beta = 4.17 \), it occurs only when \( h > 0.76 \) and \( h < 4.0 \). The second order subharmonic vibration will disappear when \( h < 1.0 \) for those systems. The phenomenon can be explained by a vibratory wave form as shown Fig.11(b). When the vibratory displacement \( x \) is less than the offset distance of the linear system \( h \), the rotor will vibrate at frequency \( \omega_{01} = (K_1 / M)^{1/2} \) and when \( x > h \), it will vibrate at \( \omega_{02} = (K_2 / M)^{1/2} \). It is the nature of nonlinear vibration that vibratory frequency changes in a vibrational “circle”. When \( h \) is very small, the rotor vibrates mainly at frequency \( \omega_2 \). With the decrease of \( h \), the effect of nonlinearity becomes smaller. So one of the phenomena of nonlinear vibration, pseudo-critical peaks (subharmonic vibration), disappears. When \( h \) becomes very large, the vibrational content at frequency \( \omega_1 \) will become predominant and subharmonic response will also be reduced.

Figure 12 generalizes the influence of nonlinearity on the subharmonic pseudo-critical peak amplitude. In order to get
high order subharmonic response, the system for a low value of damping ratio (\(\xi = 0.0056\)) is selected to be investigated. The damping is too low to suppress any pseudo—critical peaks when a rotor is accelerated from \(n\omega_0\) to \((n+1)\omega_0\). The higher order subharmonic vibration only occurs in the systems with high degree of nonlinearity. The 4th order subharmonic vibration appears when \(\beta > 5.86\) and the 5th order subharmonic vibration appears only when \(\beta > 7.7\). But some lower orders will disappear in these systems. The effect of the degree of nonlinearity \(\beta\) on pseudo—critical peaks is different for different orders. For most orders of subharmonic vibration, the maximum value of pseudo critical peaks decreases with the increase of \(\beta\). But for the 2nd order, the maximum value appears at \(\beta = 3.33\).

**EXPERIMENT**

The fundamental test rig used in this paper for a rotor system with soft nonlinear spring characteristics was well documented in 1992 (Liu and Yan) and is only mentioned here for brevity. The rigid rotor was supported on a bearing housing, which was connected with machinery case with shape memory alloy (TiNi) wire. The TiNi wire in pseudo—elastic deformation has extremely soft nonlinearity. So the rotor could be assumed as a system with two degrees of freedom and with soft nonlinear spring characteristics. The weight of equivalent mass lumped at the bearing station was 7.3kg. The supporting stiffness got very smaller when the displacement of a rotor in any direction was greater than 0.1mm approximately. The stiffness of the softer support was \(K_2 = 1.08 \times 10^7\) N/m and the harder one is \(K_1 = 3.6 \times 10^7\) N/m. Frequencies of a rotor system correspond to the two stiffnesses were 193Hz and 353Hz respectively. Figure 13 illustrates vibratory response when the rotor was accelerated and decelerated for passing through critical speeds, respectively. As shown in Fig.13, when the rotor was accelerated, the transient response bounced at a frequency of 251Hz, and its phase response was changed from 90° to 270° at same time. When the rotor was decelerated, the transient response jumped to the lower level at a frequency of 225Hz and its amplitude was larger than that when the rotor was accelerated. Its phase response returned back from 270° to 90°. When the rotor was accelerated to above 400Hz, there was occurrence of \(1/2\) subharmonic vibration. Its maximum amplitude was 0.28mm. Figure 14 shows vibratory wave forms when the experimental rotor rotated at 15120rpm and 24600rpm. It could be found that the response at 24600rpm consisted of synchronous vibration at forcing frequency and subharmonic vibration at \(1/2\) forcing frequency and the main component was \(1/2\) subharmonic vibration. So the experimental results have given a better demonstration about the nature of transient and subharmonic response for a rotor system with soft nonlinear spring characteristics.

**CONCLUSION**

The rotor with soft nonlinear spring characteristics will tend to bounce when the system is accelerated through the critical speeds and its vibratory displacement is greater than the offset distance of the linear system with stiffness \(K_1\). The vibratory motion after bouncing is predominately at the forcing frequency and slightly at its fundamental frequency. In general, the increase of damping \(\xi\), nonlinearity \(\beta\) will result in the reduction of the maximum amplitude of transient response, but there is a minimum value of the maximum amplitude in variation of the offset distance \(h\) for a fixed residual unbalance mass. The maximum amplitude is much smaller than that of the linear system with stiffness \(K_1\). So, the soft nonlinear spring characteristic can be utilized in suppressing the transient response of a rotor accelerated for passing through critical speeds. The smallest value of the maximum amplitude can be obtained by the optimal selections of the parameters \(\beta\), \(\xi\), \(h\) and \(\alpha\). But the transient response amplitude of the rotor decelerated for passing through...
critical speeds is generally greater than that of the linear system with stiffness $k_1$ and is determined by initial motion conditions. So the stiffness of a rotor should be avoided being soft nonlinear when the rotor is decelerated.

Subharmonic vibration can be experienced by a nonlinear spring rotor when it is operated beyond critical speeds and the rotor is excited by residual unbalance. The rotor will tend to bounce at its fundamental frequency when the rotor is operated at or near a speed $\frac{1}{n}$ times the bounce frequency, so that the dominant frequency of the response will be $\frac{1}{n}$ times that of the operating speed. The general nature of subharmonic vibration of a soft nonlinear spring system is very different from that of a hard one. In general, low order, (for example, 2nd order) subharmonic vibration seldom occurs and its pseudo — critical peak is very small, except great offset distance. This phenomenon turns out to be in contrary to that of a hard nonlinear spring system. For some of soft nonlinear spring systems, the vibratory response in the transition region between nth and (n+1)th order subharmonic response has not the characteristics of chaotic behavior which all hard nonlinear spring systems have, instead by synchronous vibration at the forcing frequency. In addition, the effects of offset distance $h$ and acceleration are investigated, which have never been mentioned previously but are useful in analyzing and eliminating the subharmonic vibration for rotation machinery. Some industrial rotor systems operating in bearing clearances have soft nonlinear spring characteristics only in one direction. All results reported in this paper can be directly used in analyzing the vibratory behavior of the rotor in the direction. The behavior in vertical direction is similar to that of a linear system.

ACKNOWLEDGMENT
The research project was supported by Postdoctoral Science Foundation of China.

References
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APPENDICES
Analytic Model
The system was modeled in the form of a rigid rotor mounted on a flexible housing, a single point mass with two degrees of freedom, as shown in Fig.15.
\[
\begin{align*}
M \frac{d^2}{dt^2} (X + E \cos \Omega t) + C \frac{dX}{dt} + G(X) &= 0 \\
M \frac{d^2}{dt^2} (Y + E \sin \Omega t) + C \frac{dY}{dt} + G(Y) &= 0
\end{align*}
\]  
(A1)

\[
G(X), G(Y) \text{ are the nonlinear forces in } X \text{ and } Y \text{ directions. They may be represented by the following relationships.}
\]

\[
\begin{cases}
G(X) = \begin{cases}
K_1 X & x \leq H \\
K_1 H + K_2 (X - H) & X > H
\end{cases} \\
G(Y) = \begin{cases}
K_1 Y & y \leq H \\
K_1 H + K_2 (Y - H) & Y > H
\end{cases}
\end{cases}
\]  
(A2)

The damping ratio refers to critical damping of the first linear system with the larger stiffness \(K_1\).

\[
\xi_1 = \frac{C}{2 \sqrt{K_1 M}}
\]

The degree of nonlinearity is measured by the ratio of spring stiffnesses

\[
\beta = \frac{K_1}{K_2}
\]

where, \(\beta > 1, \beta = 1, \beta < 1\) constitute a soft nonlinear system, a linear system and hard nonlinear system respectively. The system may be simplified for systematic study by the following parameter normalization. let,

\[
t = \frac{\Omega}{\Omega_1}, \quad \gamma = \frac{\Omega}{\Omega_1}, \quad x = \frac{X}{E}, \quad y = \frac{Y}{E}, \quad h = \frac{H}{E}
\]

\[
\Omega_1 = \sqrt{\frac{K_1}{M}}, \quad g(x) = \frac{G(X)}{M E \Omega_1}, \quad g(y) = \frac{G(Y)}{M E \Omega_1^2}
\]

Substituting above formulas into (A1) and (A2), the equations of motion then become

\[
\begin{align*}
\frac{d^2 x}{dt^2} + 2 \xi_1 \frac{dx}{dt} + g(x) &= \gamma^2 \cos(\gamma t) \\
\frac{d^2 y}{dt^2} + 2 \xi_1 \frac{dy}{dt} + g(y) &= \gamma^2 \sin(\gamma t)
\end{align*}
\]

and

\[
\begin{align*}
g(x) &= \begin{cases}
x & x \leq h \\
\frac{1}{\beta} (x - h) & x > h
\end{cases} \\
g(y) &= \begin{cases}
y & y \leq h \\
\frac{1}{\beta} (y - h) & y < h
\end{cases}
\end{align*}
\]

**Steady State Response**

Asymptotic method is used in deriving the formulas indicating steady state response. So the whole force is divided into a linear force and a nonlinear force. The nonlinear force can be written as

\[
\begin{align*}
g'(x) &= \begin{cases}
(1 - \frac{1}{\beta}) x & -h < x < h \\
-\frac{1}{\beta} x & x \geq h \\
(1 - \frac{1}{\beta}) x & x \leq -h
\end{cases}
\end{align*}
\]

Adding a little parameter in front of the nonlinear force and unbalance force, the equation of motion can be rewritten as,

\[
\frac{d^2 x}{dt^2} + \frac{1}{\beta} x = -g'(x) - g''(x) + \gamma^2 \cos(\gamma t)
\]

where

\[
g''(x) = 2 \xi_1 \frac{dx}{dt}, \quad \omega = \sqrt{\frac{1}{\beta}}
\]

If we assume that \(\omega - \gamma\) is very small and amplitude and phase of steady state response are the functions of \(\varepsilon\). The first asymptotic solution of steady state response can be presented as

\[
A = \frac{\gamma^2}{\sqrt{(\omega^2 (A) - \gamma^2)^2 + 4 \gamma^2 \xi_1^2 (A)}}
\]

where

\[
\begin{align*}
\omega_1 (A) &= \omega - \frac{(\beta - 1)}{2 \tan Z} (Z \sin^{-1} \frac{1}{Z} + \sqrt{1 - \frac{1}{Z^2}}) \\
\xi_1 (A) &= \xi_1, \quad Z = \frac{A}{h}
\end{align*}
\]

**Numerical Solution**

When a rotor system is accelerated, the tangential force of unbalance mass should be added to the right sides of the equations of motion. The equation is written as

\[
\begin{align*}
\frac{d^2 x}{dt^2} + 2 \xi_1 \frac{dx}{dt} + g(x) &= \gamma^2 \cos(\gamma t) + \lambda \sin(\gamma t) \\
\frac{d^2 y}{dt^2} + 2 \xi_1 \frac{dy}{dt} + g(y) &= \gamma^2 \sin(\gamma t) - \lambda \cos(\gamma t)
\end{align*}
\]

This system of equations was programed for fourth-order integration by the Runge-Kutta method in order to compute \(x\) and \(y\). The transient response is a function of \(t\) for rotational speed ratio \(\gamma\), damping ration \(\xi_1\), the degree of nonlinearity \(\beta\), and offset distance of the linear system \(h\). The computation thereby became an exercise in systematic variation in parameters. If the effect of a parameter was studied, all other parameters were fixed. When a rotor was accelerated, all computations began from 0, so all initial values were equal to zero. When it was decelerated, the computation began at a very large rotational speeds and carried out for a sufficient number of "cycles". So that initial transients were damped out and their steady state trajectory was arrived. The time increment for the integration was short enough to have no effect on computed results.