ABSTRACT

This work offers the first known three-dimensional (3-D) continuum vibration analysis for rotating, laminated composite blades. A cornerstone of this work is that the dynamical energies of the rotating blade are derived from a 3-D elasticity-based, truncated quadrangular pyramid model incorporating laminated orthotropy, full geometric nonlinearity using an updated Lagrangian formulation, and Coriolis acceleration terms. These analysis sophistications are included to accommodate the nonclassical directions of modern blade designs comprising thin, wide chord lifting surfaces of laminated composite construction. The Ritz method is used to minimize the dynamical energies with displacements approximated by mathematically complete polynomials satisfying the vanishing displacement conditions at the blade root section exactly. Several tables and graphs are presented which describe numerical convergence studies showing the validity of the assumed displacement polynomials used herein. Nondimensional frequency data is presented for various rotating, truncated quadrangular pyramids, serving as first approximations of practical blades employed in aircraft engines and fans. A wide scope of results explain the influence of a number of parameters coined to rotating, laminated composite blade dynamics, namely aspect ratio (a/b), chord ratio (c/b), thickness ratio (b/h), variable thickness distribution (h1/h2), blade pretwist angle (\( \phi_p \)), composite fiber orientation angle (\( \theta_c \)), and angular velocity (\( \Omega \)). Additional examples are given which elucidate the significance of the linear and nonlinear kinematics used in the present 3-D formulation along with the importance of the Coriolis acceleration terms included in the analysis.

INTRODUCTION

Chronicled in the published literature over the past decades are hundreds of references related to the rotating blade vibration problem [Rao (1973, 1977, 1983), Leissa (1980, 1981)]. Much of the previous work done on this problem employed one-dimensional beam models, while considering, oftentimes not collectively, various complicating topics, such as variable crosssection, pretwist, skewness, precone (sweep), curved spanwise axis, centrifugal axial stiffening and lateral softening, Coriolis force, first-order shear deformation and rotary inertia terms, combined bending and torsion, large displacements, elastic support constraints, and attached shrouds. Some of these complicating effects were considered collectively in some recent dynamical studies of composite turboprops using beam models [Komatits and Friedmann (1989), Subrahmanyan and Kaza (1986)].

It is well known that when the blade aspect ratio is small, generalized beam models for the rotating substructures in Fig. 1 are inadequate. Similar inadequacies are presented by classical plate models employing the Poisson’s hypothesis of normals to the midplane before deformation remain so after deformation. With such models, the flexural stresses are underestimated, whereas the natural frequencies are overpredicted [Dokainish and Rawtani (1971), MacBain (1975), Shaw et al. (1988)], especially in the higher modes. These results are due to the neglect of transverse shear stresses in the classical approach. It turns out that a number of investigators have exercised plate formulations which allow first-order transverse shear flexibilities to analyze the linear dynamics of rotating isotropic blades [Ramamurti and Kleib (1986)] and composite ones [Henry and Lalanne (1974), Wang et al. (1987)]. More recently, Bhumba et al. (1989, 1990) extended the analysis of shear deformable, composite rotating blades by including geometric nonlinearity in the form of von Kármán strains along with plane stress assumptions in the constitutive relations.

It turns out that the work of Bossak and Zienkiewicz (1973) provided the first known 3-D elasticity-based finite element formulation apropos to the natural vibration of centrifugally stressed solids such as those used in turbomachinery. In the dissertation of Jacob (1986), the vibration and buckling of twisted parallelepipeds were addressed using 3-D elasticity theory including differential stiffness effects due to initial stress. However, Jacob’s work did not include second order strains due to geometric nonlinearity nor did it consider dynamical energies resulting from...
centrifugal accelerations. The latter accelerations (minus those due to Coriolis acceleration) were incorporated in the 3-D vibration analysis of Galmo (1991) apropos to rotating, isotropic cantilevered blades.

We know of no previous or ongoing work offering a 3-D elasticity-based theory for vibrations of rotating laminated composite blades including full geometric nonlinearities and centrifugal accelerations in the blade kinematics. It is our purpose here to employ such theories in energy-based Ritz models of blades, so as to efficiently and accurately determine the steady-state displacement and natural vibration of arbitrarily-shaped, laminated composite blades prestressed by large displacement-dependent centrifugal loads.

The advantages of the present 3-D approach are twofold. First, no additional kinematic constraints (as in beam, plate or shell theories) are imposed, except the vanishing displacement conditions at the root section of the blade. Imposition of the latter displacement constraint is frequently a restrictive assumption model of the present Ritz approach. Second, the effects of transverse shear strains, rotary inertia, antialiastic flexure, and in-plane behaviors (such as thickness-shear and through-thickness actions) are inherent. The impact of this work provides to structural dynamists and aeroelasticians a versatile nonlinear simulation model that emits cost effective analysis experimentation of rotating blade dynamics over a wide range of geometrical and material property parameters.

Consider the cantilevered, twisted, truncated quadrangular pyramid shown in Fig. 1, as a first approximation of typical blading used in aircraft engines and fans. A truncated quadrangular pyramid is a pyramid or worm configuration having a rectangular base and an arbitrarily-oriented plane truncating the top portion of the pyramid. As shown in Fig. 1, the reference dimensions of the quadrangular pyramid are length a, root chord b, thickness hi, root thickness h1(root) at the tip. A linear variation of pretwist is assumed along the length of the pyramid, such that $\phi = \phi(a/x)$, which results in the angle of pretwist being zero at the root and $\phi$ at the tip. The truncated quadrangular pyramid has a variable chordwise width along the spanwise (x) direction, given as $w = b - (1-c/b)(x/a)$, where c/b is the chord ratio. Besides this, variable thickness is assumed in the spanwise (x) and chordwise (y) directions. The variable thickness parameters are defined in terms of the leading and trailing edges (l and t, respectively) upon which they are located (see Fig. 1). For instance, the leading and trailing edge thicknesses at the root are denoted as $h_l(root)$ and $h_t(root)$, respectively, while like thicknesses at the tip are indicated as $h_l(tip)$ and $h_t(tip)$, respectively. Variable thickness in the chordwise (y) direction is defined as $h_y = h_l(root) - (h_y(root) - h_y(tip))(y/b)/b$, where $h_y(root)$ and $h_y$ are the leading and trailing edge thicknesses, respectively, at a typical section of the truncated quadrangular pyramid. Similar variable thickness distributions are assumed in the spanwise (x) direction, which are $h_x = h_l(root) - (h_x(root) - h_x(tip))(x/a)/a$ and $h_t(root) - h_t(tip))(x/a)/a$. The present work offers the first known 3-D continuum vibration results for a rotating, laminated composite blade modeled as a cantilevered, truncated quadrangular pyramid. The dynamical energies are constructed using 3-D elasticity theory incorporating laminated orthotropy and full geometric nonlinearity using an updated Lagrangian formulation and including all centrifugal acceleration effects.

Displacements are assumed as simple polynomials, which are mathematically complete (Kantorovich and Krylov (1958)) and which satisfy the vanishing displacements at the blade root exactly. The Ritz method is used to minimize the dynamical energies to obtain upper bound approximate natural frequencies as close to the exact ones, as sufficient numbers of polynomial terms are retained. The accuracy of the present 3-D method is established by convergence studies explicitly showing the influence of solution determinant size. Nondimensional frequency results are studied showing the effect of several parameters coined to rotating, laminated composite blade vibrations, namely aspect ratio (a/b), chord ratio (c/b), thickness ratio (b/h), variable thickness distribution ($h_1(h_1)/h_t(h_t)$), blade pretwist angle ($\phi_0$), composite fiber orientation angle ($\theta$), and angular velocity ($\omega$). Additional examples are given which elucidate the significance of the linear and nonlinear kinematics used in the present 3-D formulation along with the importance of the Coriolis effects included in the analysis.

THEORETICAL FORMULATION

The strain vector ($e$) is related to the displacements $u$, $v$, and $w$ along the Cartesian coordinates $(x,y,z)$ of the blade (or truncated quadrangular pyramid as shown in Fig. 1),

$$\mathbf{e} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z, \mathbf{T}_{xy}, \mathbf{T}_{xz}, \mathbf{T}_{yz})^T = (e^T + e^{NL}) \quad (1)$$

where ($e^T$) is the linear classical strain vector and ($e^{NL}$) is the nonlinear geometric strain vector incorporating the usual squares and products of the displacement gradients, in eqn. (1).

$$\mathbf{e}_x = \partial u/\partial x + 1/2[(\partial u/\partial x)^2 + (\partial v/\partial x)^2 + (\partial w/\partial x)^2]$$

$$\mathbf{e}_y = \partial v/\partial y + 1/2[(\partial v/\partial y)^2 + (\partial w/\partial y)^2 + (\partial w/\partial y)^2]$$

$$\mathbf{e}_z = \partial w/\partial z + 1/2[(\partial w/\partial z)^2 + (\partial v/\partial z)^2 + (\partial v/\partial z)^2]$$

$$\mathbf{T}_{xy} = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z$$

$$\mathbf{T}_{xz} = \partial u/\partial x + \partial v/\partial z + \partial w/\partial y$$

$$\mathbf{T}_{yz} = \partial u/\partial y + \partial v/\partial z + \partial w/\partial x$$

For laminated composite blades, the stress vector for the $p$th ply is

$$(e^{(p)}) = [Q^{(p)}e]$$

where $[Q^{(p)}] = [Q^{(p)}]$ is a matrix of stiffness coefficients for laminated orthotropy (see Appendix A). Eqn. (2) suggests that the strains are continuous, whereas the actual stresses are discontinuous across the layer interfaces as a result of the displacement "smear" approach (that is, continuous displacements assumed across layer interfaces) employed in this analysis.

Using eqns (1)-(3), the strain energy ($U$) of the rotating composite blade model (Fig. 1) is expressed by the volume integral

$$U = 1/2 \iiint (e^{(p)})^T T(e^{(p)}) dV$$

$$= 1/2 \iiint [Q^{(p)}] L_{e^{(p)}} (e^{NL})^T T(e^{NL}) dV$$

(4)

where $[Q^{(p)}] = [Q^{(p)}] (e^T)$ and $[Q^{(p)}] = [Q^{(p)}] (e^{NL})$ are linear and nonlinear stresses in the $p$th laminate, and $dV = dx dy dz$. The above energy is decomposed into

$$U = U_L + U_P + U_{NL}$$

(5)
Specifically, when \( y \) is the pitch angle, and \( e \) is the blade sweep (or pitch angle), and \( \varepsilon \) is supplemental strain energy due to tension-flexure coupling.

The order of rotations is obtained by rotations through the Euler angles (or pitch) angle, and \( e \) is the blade sweep (or pitch angle), and \( \varepsilon \) is supplemental strain energy due to tension-flexure coupling.

The total kinetic energy \( (T) \) of the rotating composite blade model (Fig. 1) is:

\[
T = 1/2 \int \left( \rho(p)x, y, z, \epsilon \right) \cdot \dot{\epsilon} \, dv
\]  

where \( \rho(p) \) is the mass density of the \( p \)th laminate of the blade, \( q = (u, v, w)^T \) is the vector of displacements, and \( a_A \) is the acceleration vector of a typical mass point \( A \) of the blade with respect to the inertial axes \( (x, y, z) \) (see Fig. 2):

\[
a_A = a_0 + \dot{q} \times (q \times (r+\dot{\theta})) + a_c + a_{rel}
\]  

where \( a_0 \) is the centripetal acceleration of the origin of the blade coordinates \( (x, y, z) \), given by

\[
a_0 = \dot{q} \times (q \times r_0)
\]  

and \( a_c \) is the Coriolis acceleration, defined as

\[
a_c = 2\dot{q} \times r_{rel}
\]  

In eqns (7)-(9), \( \dot{q} = (\dot{q}_x, \dot{q}_y, \dot{q}_z)^T \) is the angular velocity vector, \( r_0 = (x_0, y_0, z_0)^T \) is a vector of translational offsets of the blade coordinates from the inertial coordinates, \( r = (x, y, z)^T \) is the position vector of mass point \( A \) measured from the origin of the blade coordinates \( (x, y, z) \) (see Fig. 2), and finally, \( \dot{q}_{rel} = (\dot{u}, \dot{v}, \dot{w})^T \) and \( r_{rel} = (u, v, w)^T \) are the relative acceleration and velocity vectors, respectively, where the dots (\( \dot{ } \)) indicate derivatives with respect to time \( t \). Besides this, \( (\dot{q}_x, \dot{q}_y, \dot{q}_z)^T = (0, 0, 1)^T(T_{B_x})^T(T_{B_y})^T(T_{B_z})^T \)

where the transformation between the blade coordinates \( (x, y, z) \) and the inertial coordinates \( (x, y, z) \) is obtained by rotations through the Euler angles \( \Theta_x, \Theta_y, \Theta_z \), and \( \Theta_x \) about the inertial axes (see Fig. 2). Specifically, when \( \Theta_x = 0 \), \( \Theta_x \) is the blade setting (or pitch) angle, and \( \Theta_y \) is the blade sweep (or precone). The order of rotations is \( \Theta_x, \Theta_y, \Theta_z \). In eqn (10),

\[
[T_{B_1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\Theta_y) & \sin(\Theta_y) \\ 0 & -\sin(\Theta_y) & \cos(\Theta_y) \end{bmatrix}
\]

\[
[T_{B_2}] = \begin{bmatrix} \cos(\Theta_y) & 0 & -\sin(\Theta_y) \\ 0 & 1 & 0 \\ \sin(\Theta_y) & 0 & \cos(\Theta_y) \end{bmatrix}
\]

\[
[T_{B_3}] = \begin{bmatrix} \cos(\Theta_z) & \sin(\Theta_z) & 0 \\ -\sin(\Theta_z) & \cos(\Theta_z) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Using eqns (6)-(11), several dynamical energies are employed. First, the usual kinetic energy is

\[
T_0 = \int \int \rho(p) \dot{T} \, dv
\]  

Additional kinetic energy due to centrifugal accelerations is

\[
T_\Omega = \Omega^2/2 \int \int \rho(p) \dot{T} \, dv
\]  

and supplemental energy due to Coriolis acceleration is

\[
T_C = 2\Omega \int \int \rho(p) \dot{T} \, dv
\]  

Finally, the work done by the centrifugal body forces is

\[
(U_0) = (P_0)^T \Theta_0 = \Omega^2/2 \int \int \rho(p) \dot{T} \, dv
\]  

In eqns (13)-(15), \( [Q] = [R]^T[R] \), where

\[
[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\Theta_y) & \sin(\Theta_y) \\ 0 & -\sin(\Theta_y) & \cos(\Theta_y) \end{bmatrix}
\]

In using the Ritz method, the total potential energy of the rotating blade given by

\[
x = U_L + U_p + T_0 - T_\Omega - T_C - U_0
\]  

is minimized with respect to a set of generalized coefficients \( q \) used in the Ritz trial space for \( u, v, \) and \( w \). For free vibrations, the displacement components of the rotating blade are assumed simple harmonic, as follows:

\[
u(x, y, z, t) = u_1(x, y, z) e^{i\omega t}
\]

\[
v(x, y, z, t) = v_1(x, y, z) e^{i\omega t}
\]

\[
w(x, y, z, t) = w_1(x, y, z) e^{i\omega t}
\]  

where \( \omega \) is the circular frequency of vibration, \( \varepsilon \) is the exponential function, \( t \) is time, and \( j=(-1)^{1/2} \). The displacement functions \( (u_1, v_1, w_1) \) are expressed in terms of algebraic polynomials, as follows:

\[
u_1(x, y, z) = \sum_{p=1}^{P} \sum_{q=0}^{Q} \sum_{r=0}^{R} A_{pqr} x^p y^q z^r
\]

\[
v_1(x, y, z) = \sum_{p=1}^{P} \sum_{q=0}^{Q} \sum_{r=0}^{R} B_{pqr} x^p y^q z^r
\]

\[
w_1(x, y, z) = \sum_{p=1}^{P} \sum_{q=0}^{Q} \sum_{r=0}^{R} C_{pqr} x^p y^q z^r
\]  

where from eqn (19), we define \( q = (A_{pqr}, B_{pqr}, C_{pqr}) \) as generalized coefficients to be determined. The
summation index \((p=1)\) in eqns (19) indicates that each term of the series satisfies the vanishing displacement conditions at the blade root, that is \(u_0(0,y,z) = v_0(0,y,z) = w_0(0,y,z) = 0\). Substituting eqns (18) and (19) into eqn (17), and setting the exponential terms to unity, one obtains the maximum dynamical energies, which in turn are minimized with respect to the generalized coefficients \(\{q\}\) of the trial functions \([eqns (19)]\). If no kinematic constraints other than the blade root conditions are imposed, and no other admissible terms up to \(Q\), \(R\) are omitted, then the set of polynomial terms in eqns (19) are mathematically complete \([Kantorovich and Krylov (1958)]\). With a sufficient number of terms, the calculated frequencies should in principle converge from above to exact values.

The Ritz minimization result in \(3xP(x+1)x(R+1)\) nonlinear algebraic equations, which are iteratively solved for \(\{q\}\). The nonlinear equations of motion are expressed as

\[
[K]\{\dot{q}\} + K_{CF}\{\dot{q}\} + [C]\{q\} + [M]\{\dot{q}\} = \{p_{CF}\} \tag{20}
\]

where \([K]\) contains the linear, centrifugal, and geometrically nonlinear stiffness contributions, \([C]\) and \([M]\) are the gyroscopic and mass matrices, and \([p_{CF}]\) is the centrifugal force vector. Definitions of the structural matrices and vectors in eqn (20) are given in Appendix B.

Eqn (20) is solved in two steps: first, determine the nonline static equilibrated position \(\{q_s\}\) due to the centrifugal force \([p_{CF}]\), and second, perform a free vibration analysis using the time-varying terms of eqn (20) and \(\{q_s\}\) as input. The geometrically nonlinear static analysis is performed by solving the following equation

\[
[K]_{NL}\{\dot{q}\} + [C]_{NL}\{\dot{q}\} + [M]\{\dot{q}\} = \{p_{NL}\} \tag{21}
\]

using Newton-Raphson iterative solution procedures. In the second step, the rotating blade undergoes a linear time-dependent perturbation \(\{q(t)\}\) about the static displaced position \(\{q_s\}\), as follows

\[
[K]_{T}\{\dot{q}\} + [C]\{\dot{q}\} + [M]\{\dot{q}\} = \{0\} \tag{22}
\]

where

\[
[K]_{T} = [K]_{NL} + K_{CF} \tag{23}
\]

Assuming \((x)\{q_s, q_s\}^T = \{c\} \alpha^t = \{c\} \cdot e^{i\omega t}\), a first order eigenvalue problem is applicable

\[
[A] - \lambda[B]\{x\} = \{0\} \tag{24}
\]

where \([A]\) is symmetric and \([B]\) is antisymmetric:

\[
[A] = \begin{bmatrix} [M] & [0] \\ [0] & [K]_T \end{bmatrix} \quad [B] = \begin{bmatrix} [0] & [M] \\ [-M] & [-C] \end{bmatrix} \tag{25}
\]

and \(\lambda\) is a complex eigenvalue; the real part representing exponential variations (of decay rate = \(\zeta\)) and the imaginary part as the harmonic component (at circular frequency \(w\)).

Finally, the twisted coordinates \((E, n, y)\) shown in Fig. 1 are related to the Cartesian ones \((x, y, z)\) through the orthogonal transformation

\[
\xi = x ; \quad \eta = y \cos \phi + z \sin \phi \\
\gamma = -y \sin \phi + z \cos \phi
\]

or inversely

\[
x = \xi ; \quad y = \eta \cos \phi - \gamma \sin \phi \\
z = -\eta \sin \phi + \gamma \cos \phi
\]

where \(\phi = \frac{\pi}{2}\) \((x/a) = \phi = \pi/2\). Since the transformation eqns (26) and (27) are orthogonal, the Jacobian of the transformation between \((x, y, z)\) and \((E, n, y)\) is unity (that is, \(\delta x/\delta y = -\delta z/\delta y = \delta y/\delta y = 0\)). For convenience of analysis, volumetric integral terms of the structural matrices and vectors (Appendix B) are first transformed to twisted coordinates, and then numerically integrated, since the change of variables yields integrands which are prohibitively complicated for exact integration.

**NUMERICAL STUDIES AND VALIDATION OF 3-D RESULTS**

Using the 3-D analysis outlined in the previous discussion, it is now appropriate to address how many terms of the assumed displacement polynomials \([eqn (19)]\) are required to yield reasonably accurate vibration solutions. For example, a subset of converged study of the first 10 nondimensional frequencies \(\omega a^2(p/D)^{1/2}\) of a rotating cantilevered blade with sweep modeled as a moderately thick parallelepiped with sweep \((a/b = 1, b/h = 1\) to \(10\), \(X0 = 3a\), \(y0 = 2a\) to \(0\), \(
\omega = \sqrt{k_D^2 + \omega_0^2}
\)

where \(h = 10\), \(a = 30\) \((x)\), \(y = 1\) \((y)\), \(z = 2\) \((z)\), and \(\omega\) is the natural frequency of the first mode of the stationary cantilevered parallelepiped. The blade model is assumed isotropic with Poisson's ratio \((v)\) set to 0.3 in the present 3-D formulation. Double precision arithmetic on an IBM 3090 machine has been used for all numerical calculations.

Convergence of frequency results are shown in Table 1 as the solution size is increased from a \(3x3x3\) (81 d.o.f.) solution to a \(6x6x6\) (2160 d.o.f.) solution. For instance, a \(6x6x4\) (360 d.o.f.) solution indicates that 6 terms in the \(x\)-direction, 5 terms in the \(y\)-direction, and 4 terms in the \(z\)-direction have been retained in the Ritz trial functions \([eqns (19)]\) for a total of 360 degrees of freedom (d.o.f.). A small sequential level of convergence is exhibited by the frequency solutions of Table 1, since large solution sizes contain all terms of certain of the previous smaller ones. For example, the \(5x5x4\) solution which in turn contains all terms of the \(4x4x4\) solution, and so on.

One can see clearly from the variation of \(\omega a^2(p/D)^{1/2}\) with increasing d.o.f. in Fig. 3 that a reasonably good monotonic convergence of solution for at least the first 9 frequency modes is achieved using the present 3-D Ritz analysis. Indeed, the least upper bound frequency values are obtained by using either a \(5x5x4\) or a \(6x5x4\) (360 d.o.f.) solution. The frequencies of the tenth mode do show some convergence difficulty, which is expected when using a Ritz approach. By retaining additional terms in the displacement polynomials \([eqns (19)]\), one can, in principle, achieve better convergence of solution for mode 0, however, some matrix ill-conditioning in the mass operator prohibits further reduction of the eigensystem with excessively large determinant sizes (i.e. large d.o.f.). This matrix ill-conditioning is due to the nonhomogenous nature of the algebraic polynomials assumed in eqns (19). Use of these polynomials, albeit simple, results in a mass operator characteristic of a Hilbert matrix, which can be notoriously ill-conditioned when inadequate computational precision is utilized.

Table 2 shows the convergence of the first 10 nondimensional frequencies \((\omega a^2h)/(pEz)^{1/2}\) of a rotating, cantilevered laminated composite blade modeled as a cantilevered parallelepiped having a Graphite/Epoxy laminated material with \(\{t30/T30\}\) symmetric stacking of eight layers with the thickness of each layer being 0.0052". Additional material...
properties and geometric parameters for this example blade model are defined as follows [cf. Crawley (1979), Baharlou and Leissa (1987)].

Fig. 3, the variation of $(wa_2/h)(p/E_2)^{1/2}$ effects. As can be seen in Table 3, these effects tend to destabilize the frequencies obtained from CPT, more so in the higher modes than the lower ones. Moreover, the shear correction used to overcorrect the true effect of transverse shear flexibilities in the higher modes of thick blades. In Table 3, this trend is clearly seen as one compares the slightly higher frequency values obtained by the present 3-D analysis to those frequency values calculated by the higher level of approximation used in the 8-noded Mindlin plate finite element discretizations of Bhumba, et al. (1989).

Assumed displacement models based on classical plate theory (CPT) [Leissa (1969), Baharlou and Leissa (1987)] assume that normals to the plate's undeformed midsurface remain normal after deformation. Of course, the CPT displacement models do not allow for shear deformation, although related inertia effects are many times included by investigators. In contrast, Reissner-Mindlin plate deformation models [Bhumba, et al. (1989)] assume that normals to the plate's undeformed midsurface remain straight but not necessarily normal to the midsurface after deformation. Assumed displacement models based on Mindlin's plate theory (MPT) limit their accuracy to the vibrations of moderately thick plates, mainly because the shear strain distribution of Mindlin plates is assumed to be uniform and independent of the plate's cross-sectional area.

In calculating the 3-D results in Tables 3 and 4, the in-plane and normal displacement fields are assumed as a cubic function in $z$ to yield a parabolic shear approximation for the actual nonuniform shear distribution through the plate thickness due to cross-sectional warping. Moreover, full geometric nonlinear strains were retained in conjunction with laminated orthotropicity in the 3-D constitutive laws. Assumed displacement models based on first-order shear deformable plate theory (FOSDPT) [Bhumbla, et al. (1989), Bhumbla, et al. (1989)] assumed normal displacements which were independent of the plate's thickness coordinate, resulting in zero normal strain and allowing for the usual plane stress assumptions in the constitutive relations.

Shown in Table 4 is a comparison of analytically and experimentally determined cyclic frequencies of graphite/epoxy laminated composite blades having $a/b=1$ and $a/b=2$ and various ply stacking sequences. Generally speaking, the agreement between the analytical results is favorable in most of the frequency modes. In fact, the upper bound frequencies calculated by the present 3-D method are lower than the finite element frequencies reported by Crawley (1979) by about 2 percent in some of the higher modes, largely due to the inherent shear deformation and rotary inertia present. Across the board it does appear that the experimental data reported by Crawley (1979) is lower than the analytical results, albeit the present 3-D frequencies do agree more favorably with the experimental ones, particularly in the higher modes. Crawley offered that a 20 percent reduction in the transverse shear modulus used to calculate his finite element results in a less than one percent reduction in frequency values. This led to a conclusion that the transverse shear effects was not the cause of the difference between the analytical and experimental cyclic frequencies shown in Table 4. Given the close agreement between the analytical and experimental results in Table 4, it is difficult to make a clear conjecture about a primary cause of some of the destabilizing present in the experimental frequencies.
From the data shown in Tables 5 and 6, one can fully ascertain what is the influence of the geometrically nonlinear strains and Coriolis acceleration terms retained in the present 3-D analysis of the natural vibration of typical blading used in practical disk assemblies. Table 5 list nondimensional frequencies \( \omega^2 (\rho h g D) / h^2 \gamma \) of a rotating twisted blade modeled as a twisted, truncated quadrangular pyramid (a/b=1, c/b=1.25, h/h_0=1.15, y_0=3a, y_0=1z_0=0, \gamma_0=0, \theta_0=45^\circ, \phi_0=0^\circ, v=0.3). Table 6 shows nondimensional frequencies \( \omega^2 (\rho h g D) / h^2 \gamma \) of a typical rotating, laminated composite, wide chord, fan blade modeled as a cantilevered, truncated quadrangular pyramid (a/b=1, c/b=1.25, h/h_0=3a, y_0=3a, y_0=1z_0=0, \gamma_0=0, \theta_0=45^\circ, \phi_0=0^\circ, v=0.3). As in Table 3, calculated frequency data obtained using a classical thin plate Ritz approach is well within the small scope of previously reported frequency data for rotating laminated composite blades have been summarized to ascertain the number of polynomial terms required to achieve a reasonable degree of accuracy of solution. One such group of frequency data obtained using a classical thin plate Ritz approach is shown in Table 7. The data in Table 7 shows a comparison of nondimensional frequencies \( \omega^2 (\rho h g D) / h^2 \gamma \) of rotating cantilevered blades modeled as moderately thick parallelepipeds (a/b=c/b=3, b/h=10, h/h_0=1z_0=0, y_0=3a, y_0=1z_0=0, \gamma_0=0, \theta_0=45^\circ, \phi_0=0^\circ, v=0.3). As in Table 3, calculated frequency data obtained using the present 3-D method (including nonlinear strains \( \epsilon_{\text{NL}} \) and Coriolis accelerations) are compared to data previously reported by Bhumbla, et al. (1989) using a FOSDPT approach. Although the agreement between the frequency results is favorable, the shear correction factor used in the FOSDPT analysis corrects to some extent by the present method, since the shear deformable plate finite element approach with linear strains \( \epsilon_{\text{lin}} \) and Coriolis acceleration terms \( \omega^2 (\rho h g D) / h^2 \gamma \) are well within the small scope of previously reported frequency data for rotating laminated composite blades have been summarized to ascertain the number of polynomial terms required to achieve a reasonable degree of accuracy of solution. One such group of frequency data obtained using a classical thin plate Ritz approach is shown in Table 7.

### CONCLUDING REMARKS

This work offers the first known 3-D continuum vibration solutions for rotating laminated composite blades. The Ritz method has been employed to determine approximate nondimensional frequencies. The dynamical energies of the rotating blade have been derived from a 3-D elasticity-based, truncated quadrangular pyramid model incorporating laminated orthotropy, full geometric nonlinearity using an updated Lagrangian formulation, and complete centrifugal and Coriolis acceleration terms. A set of algebraic polynomials, which are mathematically complete, have been utilized as trial functions. No additional kinematic constraints (such as thickness-shear and thickness-twist actions) are required to achieve a reasonable degree of accuracy of solution. One such group of frequency data obtained using a classical thin plate Ritz approach is shown in Table 7.

### ACKNOWLEDGMENTS

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### REFERENCES


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Rotating Cantilevered Truncated Pyramids M.S. Thesis, Ohio State University.


Jacob, K., 1986, Three-Dimensional Vibration and Buckling Analysis of Twisted Parallel-Edged, Ph.D. Dissertation, Ohio State University.


APPENDIX A. CONSTITUTIVE RELATIONS OF TRUNCATED PYRAMID

For laminated orthotropy, the constitutive eqn (3) is defined as

\[
\begin{pmatrix}
\sigma_{xx}^{(p)} \\
\sigma_{yy}^{(p)} \\
\sigma_{zz}^{(p)} \\
\tau_{xz}^{(p)} \\
\tau_{za}^{(p)} \\
\tau_{za}^{(p)}
\end{pmatrix} = \begin{pmatrix}
Q_{11}^{(p)} & Q_{12}^{(p)} & Q_{13}^{(p)} & 0 & 0 & 0 \\
Q_{12}^{(p)} & Q_{22}^{(p)} & Q_{23}^{(p)} & 0 & 0 & 0 \\
Q_{13}^{(p)} & Q_{23}^{(p)} & Q_{33}^{(p)} & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{44}^{(p)} & 0 & 0 \\
0 & 0 & 0 & 0 & Q_{45}^{(p)} & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{66}^{(p)}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{xx}^{(p)} \\
\epsilon_{yy}^{(p)} \\
\epsilon_{zz}^{(p)} \\
\gamma_{xz}^{(p)} \\
\gamma_{za}^{(p)} \\
\gamma_{za}^{(p)}
\end{pmatrix}
\]

where \(Q_{ij}(i,j=1,2,\ldots,6)\) are defined as

\[
Q_{11} = m_1^2 \hat{Q}_{11} + n_1^2 \hat{Q}_{12} + 2m_1n_1 \hat{Q}_{12} (\hat{Q}_{12} + 2 \hat{Q}_{66})
\]

\[
Q_{12} = (m_1 + n_1) \hat{Q}_{12} + m_1^2 \hat{Q}_{11} + n_1^2 \hat{Q}_{22} - 4 \hat{Q}_{66}
\]

\[
Q_{22} = m_1^2 \hat{Q}_{22} + n_1^2 \hat{Q}_{11} + 2m_1n_1 \hat{Q}_{12} + 2 \hat{Q}_{66}
\]

\[
Q_{16} = mn (m_1^2 \hat{Q}_{11} + n_1^2 \hat{Q}_{22} - 2m_1n_1 \hat{Q}_{12} + 2 \hat{Q}_{66})
\]

\[
Q_{26} = mn (m_1^2 \hat{Q}_{11} + n_1^2 \hat{Q}_{22} + n_1^2 \hat{Q}_{12} + 2 \hat{Q}_{66})
\]

\[
Q_{13} = m_1^2 \hat{Q}_{13} + n_1^2 \hat{Q}_{23}
\]

\[
Q_{23} = m_1^2 \hat{Q}_{23}
\]

\[
Q_{44} = m_n (m_1^2 \hat{Q}_{13} - n_1^2 \hat{Q}_{23})
\]

\[
Q_{45} = mn (m_1^2 \hat{Q}_{13} - n_1^2 \hat{Q}_{23})
\]

\[
Q_{66} = mn (m_1^2 \hat{Q}_{13} + n_1^2 \hat{Q}_{23} + \hat{Q}_{13} + 2 \hat{Q}_{12})
\]

APPENDIX B. DEFINITION OF STRUCTURAL MATRICES AND VECTORS

The following are definitions of the structural matrices and vectors used in eqn (20).

Definition of Linear Stiffness Matrix [KL]

\[
[K_L] = \begin{pmatrix}
K_{11} & K_{12} & K_{13} \\
K_{12} & K_{22} & K_{23} \\
K_{13} & K_{23} & K_{33}
\end{pmatrix},
\]

Definition of Mass Matrix [M]

\[
[M] = \begin{pmatrix}
M_{11} & 0 & 0 \\
0 & M_{22} & 0 \\
0 & 0 & M_{33}
\end{pmatrix}
\]

\[
[M^{(p)}] = \int \int \int \rho(p) \omega_i \omega_j dV,
\]

\[
[M^{(3)}] = \int \int \int \rho(p) v_i v_j dV.
\]
Definition of Centrifugal Stiffness Matrix \([K_c]\)

\[
[K_c] = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{12} & T_{22} & T_{23} \\
T_{13} & T_{23} & T_{33}
\end{bmatrix}
\]

where

\[
(T_{1i})_{ij} = \int \int \rho \left( \frac{\partial \ddot{q}}{\partial x_i} \right) \dot{q}_j \, dx \, dy
\]

(\(T_{2i}\))_{ij} = \int \int \rho \left( \frac{\partial \ddot{q}}{\partial y_i} \right) \dot{q}_j \, dx \, dy

(\(T_{3i}\))_{ij} = \int \int \rho \left( \frac{\partial \ddot{q}}{\partial z_i} \right) \dot{q}_j \, dx \, dy

Definition of Coriolis Matrix \([C]\)

\[
[C] = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{12} & C_{22} & C_{23} \\
C_{13} & C_{23} & C_{33}
\end{bmatrix}
\]

where

\[
(C_{1i})_{ij} = 2 \int \int \rho \left( \frac{\partial \ddot{q}}{\partial x_i} \right) \dot{q}_j \, dx \, dy
\]

(\(C_{2i}\))_{ij} = 2 \int \int \rho \left( \frac{\partial \ddot{q}}{\partial y_i} \right) \dot{q}_j \, dx \, dy

(\(C_{3i}\))_{ij} = 2 \int \int \rho \left( \frac{\partial \ddot{q}}{\partial z_i} \right) \dot{q}_j \, dx \, dy

Definition of Centrifugal Force Vector \([P_{CF}]\)

\[
(P_{CF}) = (p_1, p_2, p_3)^T
\]

(\(p_1\)) = \int \int \rho \left( \frac{\partial \ddot{q}}{\partial x} \right) \dot{q} \, dx \, dy

(\(p_2\)) = \int \int \rho \left( \frac{\partial \ddot{q}}{\partial y} \right) \dot{q} \, dx \, dy

(\(p_3\)) = \int \int \rho \left( \frac{\partial \ddot{q}}{\partial z} \right) \dot{q} \, dx \, dy

Definition of Nonlinear Stiffness Matrix \([K_{NL}]\)

\[
[K_{NL}] = \begin{bmatrix}
K_{11} & 0 & 0 \\
0 & K_{22} & 0 \\
0 & 0 & K_{33}
\end{bmatrix}
\]

(\(K_{11}\))_{ij} = \int \int \rho \dot{q}_i \dot{q}_j \, dx \, dy

(\(K_{22}\))_{ij} = \int \int \rho \dot{q}_i \dot{q}_j \, dx \, dy

(\(K_{33}\))_{ij} = \int \int \rho \dot{q}_i \dot{q}_j \, dx \, dy

The integration of each term in the above equations is carried out by taking the following typical form:

\[
J = \int \int \rho \dot{q} \, dx \, dy
\]

For laminated plates, this integration can be rewritten as

\[
J = \sum_{p=1}^{m} \int \int \rho \dot{q} \, dx \, dy
\]

where \(m\) is the total number of layers in a laminated configuration.
Fig. 2. Euler Angles and translational offsets of rotating blade local (x,y,z) axes with respect to inertial (x̄,ŷ,ẑ) axes

Fig. 3. Convergence of nondimensional frequencies of a rotating isotropic blade with sweep

Table 1. Convergence of nondimensional frequencies $\omega^2/(bh/\pi)^{1/2}$ of a rotating isotropic blade with sweep $(a/b=c/b=1, b/h=10, x_0=3a, y_0=z_0=0, \Omega h_0, \beta_x=30^\circ, \beta_y=0, \beta_z=0, \omega=0.3)$

<table>
<thead>
<tr>
<th>Size</th>
<th>3x3x3</th>
<th>4x4x3</th>
<th>4x4x4</th>
<th>5x4x4</th>
<th>5x5x4</th>
<th>6x4x4</th>
</tr>
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<tbody>
<tr>
<td>d.o.f</td>
<td>81</td>
<td>144</td>
<td>162</td>
<td>200</td>
<td>300</td>
<td>360</td>
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<tr>
<td>Mode 1</td>
<td>2.6420</td>
<td>2.6268</td>
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<td>3.6478</td>
<td>3.5506</td>
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<td>3.5224</td>
<td>3.5132</td>
</tr>
<tr>
<td>Mode 3</td>
<td>7.2629</td>
<td>7.0231</td>
<td>7.0230</td>
<td>6.9523</td>
<td>6.9824</td>
<td>6.9831</td>
</tr>
<tr>
<td>Mode 6</td>
<td>12.835</td>
<td>10.660</td>
<td>10.557</td>
<td>10.512</td>
<td>10.531</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Convergence of nondimensional frequencies $\omega^2/(bh/\pi)^{1/2}$ of a rotating graphite/epoxy laminated composite blade $(a=3'', b=3'', x_0=2a, y_0=z_0=0, \Omega h_0, \beta_x=45^\circ, \beta_y=0, \beta_z=13^\circ)$

<table>
<thead>
<tr>
<th>Size</th>
<th>3x3x3</th>
<th>4x4x3</th>
<th>4x4x4</th>
<th>5x5x4</th>
<th>5x6x4</th>
<th>6x5x4</th>
</tr>
</thead>
<tbody>
<tr>
<td>d.o.f</td>
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<td>144</td>
<td>162</td>
<td>200</td>
<td>300</td>
<td>360</td>
</tr>
<tr>
<td>Mode 1</td>
<td>5.3796</td>
<td>5.3251</td>
<td>5.3245</td>
<td>5.3017</td>
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<tr>
<td>Mode 2</td>
<td>7.1235</td>
<td>6.7748</td>
<td>6.7703</td>
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<td>Mode 3</td>
<td>11.615</td>
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</tr>
<tr>
<td>Mode 4</td>
<td>28.278</td>
<td>20.320</td>
<td>20.302</td>
<td>19.745</td>
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<tr>
<td>Mode 5</td>
<td>34.804</td>
<td>21.682</td>
<td>21.638</td>
<td>21.768</td>
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</tr>
<tr>
<td>Mode 6</td>
<td>53.268</td>
<td>27.133</td>
<td>27.098</td>
<td>26.724</td>
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<td></td>
</tr>
<tr>
<td>Mode 7</td>
<td>108.990</td>
<td>38.977</td>
<td>38.860</td>
<td>35.375</td>
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<td>167.400</td>
<td>61.533</td>
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<td>47.845</td>
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<tr>
<td>Mode 9</td>
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<td>86.996</td>
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<td>Mode 10</td>
<td>190.480</td>
<td>92.690</td>
<td>92.214</td>
<td>56.398</td>
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<td></td>
</tr>
</tbody>
</table>
of an isotropic cantilevered blade (a/b=c/b=1, b/h=10, \(\nu=0.3\))

Bhumbla, et al. (1989)

Table 4: Comparison of cyclic frequencies (Hz) of cantilevered blade (a/b=c/b=1, b/h=10, \(\nu=0.3\))

<table>
<thead>
<tr>
<th>Mode Number</th>
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<th>4</th>
<th>5</th>
</tr>
</thead>
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<td>3.4493</td>
<td>8.0958</td>
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<td>28.408</td>
</tr>
</tbody>
</table>


Table 5: Comparison of cyclic frequencies (Hz) of cantilevered blade (a/b=c/b=1, b/h=10, \(\nu=0.3\))

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present 3-D analysis</td>
<td>26.178</td>
<td>363.04</td>
<td>759.12</td>
<td>1597.5</td>
<td>1628.0</td>
</tr>
<tr>
<td>Crawley (1979)</td>
<td>234.2</td>
<td>362.</td>
<td>728.3</td>
<td>1449</td>
<td>1503.</td>
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<td>Crawley (1979)</td>
<td>261.9</td>
<td>363.5</td>
<td>761.8</td>
<td>1662</td>
<td>1709.</td>
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<tr>
<td>Bhumbla &amp; Leissa (1989)</td>
<td>262.1</td>
<td>363.7</td>
<td>771.4</td>
<td>1642</td>
<td>1653.</td>
</tr>
<tr>
<td>Leissa (1969)</td>
<td>53.8</td>
<td>46.1</td>
<td>50.4</td>
<td>54.6</td>
<td>65.7</td>
</tr>
</tbody>
</table>


Table 6: Comparison of cyclic frequencies (Hz) of cantilevered blade (a/b=c/b=1, b/h=10, \(\nu=0.3\))

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present 3-D analysis</td>
<td>55.580</td>
<td>174.78</td>
<td>344.83</td>
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<td>810.95</td>
</tr>
<tr>
<td>Crawley (1979)</td>
<td>48.6</td>
<td>169.0</td>
<td>303</td>
<td>554.</td>
<td>739.</td>
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<tr>
<td>Crawley (1979)</td>
<td>55.580</td>
<td>175.4</td>
<td>345.3</td>
<td>591.8</td>
<td>820.1</td>
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<tr>
<td>Crawley (1979)</td>
<td>138.36</td>
<td>495.45</td>
<td>795.22</td>
<td>1307.9</td>
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<td>Crawley (1979)</td>
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<td>Leissa (1969)</td>
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<td>546.0</td>
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<tr>
<td>Leissa (1969)</td>
<td>53.9</td>
<td>148.0</td>
<td>362.7</td>
<td>508.0</td>
<td>546.0</td>
</tr>
</tbody>
</table>


Table 7: Comparison of cyclic frequencies (Hz) of cantilevered blade (a/b=c/b=1, b/h=10, \(\nu=0.3\))

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>1</th>
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<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOSDPT1</td>
<td>3.5949</td>
<td>8.1779</td>
<td>20.287</td>
<td>--</td>
<td>25.651</td>
<td>28.481</td>
</tr>
<tr>
<td>FOSDPT1</td>
<td>4.0491</td>
<td>8.5289</td>
<td>20.704</td>
<td>--</td>
<td>25.873</td>
<td>28.832</td>
</tr>
<tr>
<td>FOSDPT1</td>
<td>4.6921</td>
<td>9.0653</td>
<td>22.054</td>
<td>--</td>
<td>26.238</td>
<td>29.478</td>
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<tr>
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<td>27.040</td>
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<td>6.4216</td>
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<td>32.701</td>
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