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A NEW PROBABILISTIC APPROACH FOR ACCURATE FATIGUE DATA ANALYSIS OF CERAMIC MATERIALS

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ABSTRACT

Statistical methods for the design of ceramic components for time-dependent failure modes have been developed which can significantly enhance component reliability, reduce baseline data generation costs, and lead to more accurate estimates of slow crack growth (SCG) parameters. These methods are incorporated into the AlliedSignal Engines CERAMIC and ERICA computer codes. Use of the codes facilitates generation of material strength parameters and SCG parameters simultaneously, by pooling fast fracture data from specimens that are of different sizes, or stressed by different loading conditions, with data derived from static fatigue experiments. The codes also include approaches to calculation of confidence bounds for the Weibull and SCG parameters of censored data and for the predicted reliability of ceramic components.

This paper presents a summary of this new fatigue data analysis technique and an example demonstrating the capabilities of the codes with respect to time-dependent failure modes. This work was sponsored by the U.S. Department of Energy Oak Ridge National Laboratory (DoE/ORNL) under Contract No. DE-AC05-84OR21400.

NOMENCLATURE

A	Slow crack growth parameter
a	Crack length
ANSYS	Finite element modeling computer code
ASME	American Society of Mechanical Engineers
ASTM	American Society for Testing and Materials
B	Slow crack growth parameter
°C	Degrees Celsius
CERAMIC	AlliedSignal ceramic life prediction code
DoE	U.S. Dept. of Energy
ERICA	AlliedSignal ceramic life prediction code
F, F	Probability
f	Probability density for strength observation
f	Probability density for time observation
°F	Degrees Fahrenheit
I	Multiaxial stress factors for fast fracture
I_t	Multiaxial stress factors for slow crack growth
K	Stress intensity factor
K_{Ic}	Critical mode I stress intensity factor
ksi	Thousands of Pounds Per Square Inch
$m, \hat{m}$	First Weibull parameter or Weibull slope
MPa	MegaPascals
n	Slow crack growth parameter
ORNL	Oak Ridge National Laboratory
P	Probability
P_f	Probability of failure
R	Reliability or probability of success
S	Probability of survival for strength observation
S	Probability of survival for time observation
SCG	Subcritical crack growth
t	Time
t_f	Time to failure
U.S., USA	United States of America
V	Physical size of a specimen or component
$W(\cdot)$	Likelihood ratio statistic
x_i	Observed strength
Y	Crack geometry factor
y_i	Observed time
σ_e, σ_L	Effective stress levels
σ_{max}	Maximum applied principal stress on the component
$\sigma_O, \hat{\sigma}_O$	Second Weibull parameter
ϕ, ψ	Angles used in integration of unit radius sphere

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1.0 INTRODUCTION

The design of ceramic components for structural applications can be very challenging. Often ceramic components are expected to exceed reliability standards which the metallic components they are replacing could not meet. In the process of assessing the reliability of ceramic components, two issues must be addressed: 1) Generation of the Weibull and fatigue parameters of the material, and 2) Risk integration of the component to determine the reliability. The basic theories that address these two issues are well developed (Weibull, 1939; Batdorf and Heinisch, 1978; Lamon and Evans, 1983), but in order to produce reliable designs with the typically small material properties databases and large extrapolations, more advanced statistical methods are required.

To this effect, AlliedSignal Engines has been pursuing the development of probabilistic and statistical methods for extracting the most information out of a given set of data. These efforts have been performed under the Life Prediction Methodologies for Ceramic Components of Advanced Heat Engines program, Phase I and Phase II, funded by the Department of Energy/Oak Ridge National Laboratory (DoE/ORNL) under Contract No. DE-AC05-84OR21400. The result has been the development of two computer codes, CERAMIC and ERICA, that are intended to be used in tandem (Schenk, et. al., 1998). These codes incorporate state-of-the-art methodologies for the design and life prediction of ceramic components, as follows.

First, the statistical methodologies consist of approaches that use censored data analysis techniques for the pooling of material data. By pooling, it is meant that specimens of different sizes that are loaded under different conditions (including proof testing) and at different temperatures can be analyzed together, in a single analysis, to generate the required Weibull parameters. The material parameters generated in this way are more accurate than those obtained from the individual analysis of each data set. The pooling of specimens of different sizes and loading conditions can be used to provide an indication as to how well the chosen failure theory fits the given material. This approach has been extended to allow the pooling of fast fracture and static fatigue data in order to generate more accurate estimates of the slow crack growth parameters of the ceramic material.

Second, the methodology includes approaches for the calculation of confidence bounds for the reliability prediction of a component. This is particularly important, because actual components are generally much larger and are loaded at significantly lower stress levels than test specimens, resulting in sizable extrapolations of the material properties. For actual components, the ratio of failures that originate from volume and surface flaws may also be much different than the ratio observed in the database specimens. Extrapolation away from the bulk of the stress levels where the data are generated results in significantly wide confidence bounds. One should, therefore, design ceramic components to a desired level of reliability using the upper-bound confidence limit. This methodology has been expanded to the extent of calculation of confidence bounds for the reliability of a specimen (or component) for a given lifetime and multiple, independent (but concurrently acting), flaw populations.

Third, the risk integration approach has been modified to facilitate confidence bound calculations for components. The procedure consists of the evaluation of the effective size of a component as a function of the Weibull slope, slow crack growth parameters, and time. This information is then used to compute confidence bounds for the given component.

The motivation for the development of this fatigue data analysis methodology has been the shortfalls seen in ordinary statistical treatment of fatigue data with respect to runouts and failures on loading (which are especially typical during fatigue testing of ceramic materials). A runout is defined as a test which is interrupted before the specimen fails. Runouts do not contain the same information about the placement on the stress vs. time curve as failures do, and ignoring this fact leads to serious errors in the estimated materials response.

To illustrate this, consider the following thought experiment: A test is performed very near the runout strength of a material and it does not fail before x hours. A similar test, performed at 100 MPa below "runout", is also stopped after x hours. An ordinary least squares regression of all the data, including the first example, but not the second, would have little effect on the final position of the curve. But including the very low point - as if it were a failure - will greatly lower the resulting curve.

Conventional least squares regression is the accepted method for estimating fatigue parameters of the stress-time curve only when all fatigue specimens really do fail. If runouts (and failures on loading) have to be included in the data analysis, censored data analysis using the maximum likelihood approach should be used to extract the most information out of the data available. This short introduction is illustrated in Figure 1, comparing fatigue curves derived from conventional least squares regression and the newly-developed censored maximum likelihood data analyses.

2.0 DERIVATION OF THE STATISTICAL METHOD

If stress rupture failures are a consequence of subcritical crack growth (SCG), and if SCG and fast fracture both occur from the same flaw population, fast fracture strength and stress rupture data may be pooled to perform a combined likelihood analysis of the complete fast fracture and stress rupture data. One implication from such an analysis is that the number of observations is significantly increased which in turn greatly increases the confidence of subsequent predictions.

2.1 Development Of The Likelihood Function

Development of the likelihood function which forms the basis of the analysis method requires the distribution function for the observations. The solution for the distribution of observations having strength degradation in the presence of slow crack growth under static loading and competing failure modes is derived for the situation in which each specimen is tested to failure or the test is terminated. In this case, an observation is either the strength value at failure (observation or censored observation), or either the time at which failure occurs (observation or censored observation) or the runout time. The former situation happens when failure occurs during loading to the static value, and the latter occurs when the static load has been safely reached and failure occurs after sufficient subcritical crack growth has taken place or the test is terminated with a time runout. The development is based on the multiaxial Weibull setup with assumed coplanar crack growth. This setup and assumptions give a first step in the development of analysis procedures including likelihood ratio confidence bounds that allow for subcritical crack growth in the presence of competing failure modes.

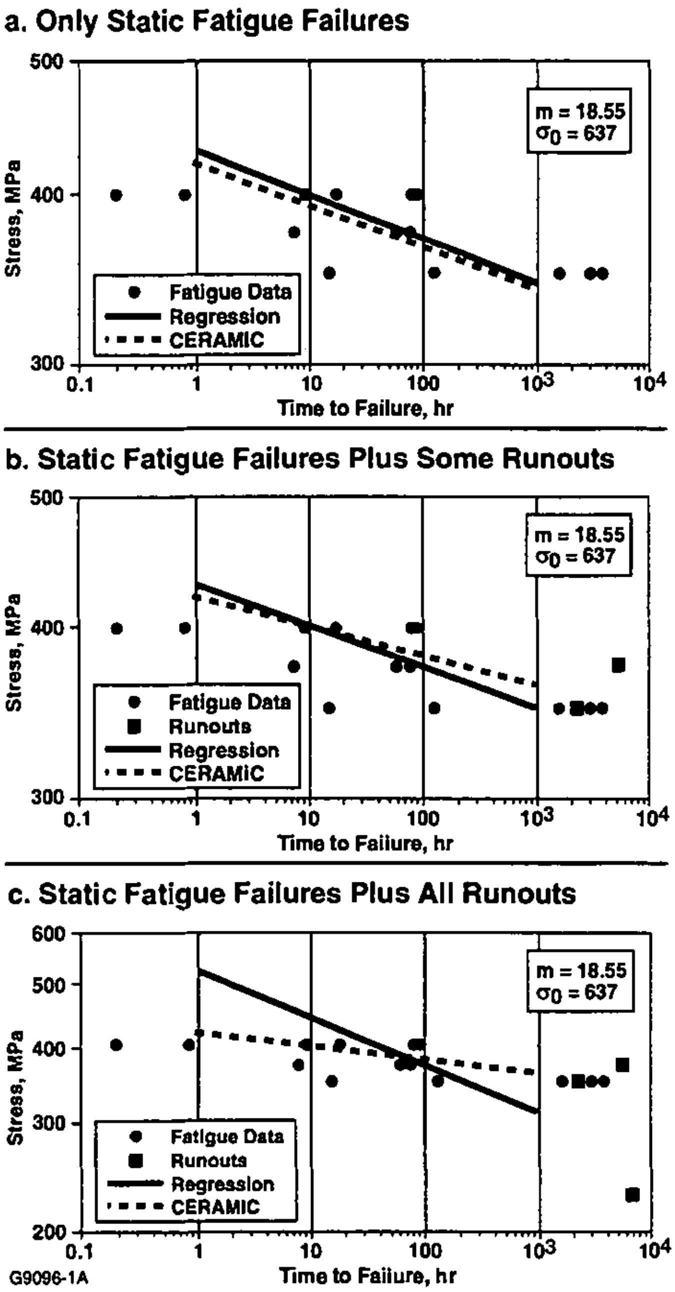

Figure 1. Conventional Fatigue Data Analysis Is Not Capable Of Accounting For Runouts Correctly

2.1.1 Transformation Of Strength Due To Slow Crack Growth

The multiaxial setup was first presented by Batdorf and Heinisch (1978) and Lamon and Evans (1983). This approach has been strengthened and generalized by Cuccio, et. al. (1994), Nemeth, et. al. (1994), Johnson and Tucker (1992, 1994), and Schenk, et. al. (1998). Also, Tucker and Johnson (1994) have shown that the Batdorf and Heinisch, and Lamon and Evans approaches are equivalent (cf. references in Tucker and Johnson). In the multiaxial setup, the probability of fast fracture failure is given by Equation [1]:

$$P_f(\sigma_{max}) = 1-\exp\left\{-IV\left(\frac{\sigma_{max}}{\sigma_0}\right)^m\right\}, \qquad [1]$$

Where: σ_{max} typically is the maximum principal stress in the component, and is strictly used for normalization purposes; m is the first Weibull parameter or Weibull modulus; σ_0 is the second Weibull parameter or characteristic strength; V is the physical size of a specimen; and I is the multiaxial stress factor (for a volume flaw population in this case), as defined by Equation [2]:

$$I = \frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\sigma_e(x,y,z,\phi,\psi)}{\sigma_{max}}\right)^m \cos\phi\, d\phi\, d\psi\, dV \qquad [2]$$

Where: σ_e is the equivalent stress determined by a suitable fracture criterion and is a value of σ_c, the critical stress, i.e., the remote stress applied normal to the crack plane that would produce failure. Similar expressions have been derived for surface and chamfer flaw populations (see Schenk, et. al., 1998). Note that σ_{max} is independent of any specific failure mode with its associated σ_e. Moreover, for a fixed loading geometry, σ_e / σ_{max} is not a function of the failure stress. Indeed, σ_e can be expressed as Equation [3]:

$$\sigma_e = \sigma_1 F[\cdot], \qquad [3]$$

Where (cf. Lamon and Evans and Tucker and Johnson), $F[\cdot]$ is not a function of σ_1, the maximum principal stress. Thus σ_{max} (the failure stress) is just the maximum value of σ_1 in the specimen. In treating time-dependent phenomena, the distribution of σ_c must be generalized to cover the change of strength that occurs over time.

The following discussion will demonstrate the approach, using the simplest form of the fracture mechanics description of SCG (the power law), Equation [4]:

$$da/dt = A(K)^n, \qquad [4]$$

Where: A and n are constants, and (Equation [5]):

$$K = Y\sigma\sqrt{a}, \qquad [5]$$

and where: Y denotes a geometry factor, σ is the applied load (stress), and a is the crack length. Then, as shown in Trantina and Johnson (1983), under uniform loading Equation [5] can be substituted into Equation [4], and Equation [4] can be integrated and rearranged to yield Equation [6]:

$$(\sqrt{a_f})^{2-n} = (\sqrt{a_i})^{2-n} + \left(\frac{2-n}{2}\right)A(Y\sigma)^n t, \qquad [6]$$

Where: $n \neq 2$, a_f denotes the final crack length $(\geq a_i)$, a_i denotes the initial crack length, and t is the time span over which the constant load, σ, is applied.

In view of the assumptions and the fact that the setup leading to Equation [6] is essentially uniaxial, Equation [6] can be used to transform an initial strength to a final strength at which failure occurs after the time span, t. In order to do this so as to determine the distribution that results from SCG, a substitution in Equation [6] for the crack lengths is made in terms of the critical stress via the relationship to K_{Ic} given by $K_{Ic} = Y\sigma_c\sqrt{a}$, where σ_c is the critical stress that will just produce failure for the length a. When this substitution is carried out and some rearrangement made, we obtain Equation [7]:

$$(\sigma_f)^{n-2} = (\sigma_i)^{n-2} + \left(\frac{2-n}{2}\right)\left(\frac{K_{Ic}}{Y}\right)^{n-2} A(Y\sigma)^n t, \qquad [7]$$

Where: $n > 2$ and the f and i subscripts denote the final and initial critical stresses, respectively. Equation [7] gives the degradation in strength for a particular failure mode, since σ_f and σ_i are values of σ_c: It is obvious that $\sigma_f < \sigma_i$ when t is greater than zero. Thus, the difference between σ_f and σ_i produces the amount of time for SCG. Also, SCG can occur if and only if $\sigma < \sigma_i$, otherwise failure occurs on applying the load, σ.

Sturmer, et. al. (1993) generalize the results of Equations [4] through [7] to the multiaxial case by rewriting Equation [5] as follows (Equation [8]):

$$K_{Ic} = Y\sigma_e(t)\sqrt{a(t)}, \qquad [8]$$

Where: $\sigma_e(t)$ is the load at time t expressed as an equivalent stress that would just produce failure for the length $a(t)$, which is changing over time. This implies, among other things, that $\sigma_e(t)$ accounts for any initial non-coplanar crack growth. Since a crack must grow in order to have fracture failure, the implication is very reasonable. Equation [8] is then substituted into Equation [4] and the integration of Equation [6] is carried out to yield Equation [9], upon the substitution of Equation [5]:

$$(\sigma_{ef})^{n-2} = (\sigma_{ei})^{n-2} + \left(\frac{2-n}{2}\right)\left(\frac{K_{Ic}}{Y}\right)^{n-2} AY^n \int_{t=0}^{t_f} (\sigma_e(t))^n dt, \qquad [9]$$

Where: $\sigma_e(t)$ is such that Equations [8] and [5] are first met at t_f, and σ_{ef} and σ_{ei} are the final and initial equivalent stresses, respectively. When the load is constant as in stress rupture tests, the integration on the right-hand side of Equation [9] can be carried out to yield equation [10]:

$$(\sigma_{ef})^{n-2} = (\sigma_{ei})^{n-2} + \left(\frac{2-n}{2}\right)\left(\frac{K_{Ic}}{Y}\right)^{n-2} AY^n \sigma_L^n t, \qquad [10]$$

Where: σ_L is the equivalent stress resulting from the constant load. Now, on failure σ_{ef} equals σ_L. Taking this into account and solving Equation [10] for σ_{ei} yields Equation [11] (dropping the subscript i and in what follows, understanding that σ_e is the initial equivalent stress):

$$\sigma_e = \sigma_L\left[1 + \left(\frac{n-2}{2}\right)(K_{Ic})^{n-2} AY^2\sigma_L^2 t\right]^{\frac{1}{n-2}} = \sigma_L\left[1 + \frac{\sigma_L^2}{B}t\right]^{\frac{1}{n-2}}, \qquad [11]$$

Where:

$$B = \left[\left(\frac{n-2}{2}\right)(K_{Ic})^{n-2} AY^2\right]^{-1} \qquad [11a]$$

Equation [10] also can be solved for t, yielding the time that is required for the initial strength to be degraded to the load and hence failure then occurs, which is (Equation [12]):

$$t = \frac{B}{\sigma_L^2}\left[\left(\frac{\sigma_e}{\sigma_L}\right)^{n-2} - 1\right]. \qquad [12]$$

It is clear that Equations [11] and [12] are in a one-to-one and onto relationship and, hence, inverses of each other. Equation [12] has other important properties; in view of Equation [3], the ratio inside the brackets in Equation [12] is a scalar ratio of the initial failure stress to that of the load failure stress and is just proportional to σ_c. Thus, except for the σ_L^2 term, t is independent of location and angle and, hence, of any element. Moreover, for a given location and angle, Equation [12] is a strictly monotonically increasing function of the scalar ratio.

Equations [11] and [12] form the basis for the development of the distribution of an observed time for a specific failure mode. Using Equation [11], Equation [2] can be rewritten to define a multiaxial stress factor, I_t for time-dependent failure modes as in Equation [13]:

$$I_t(m,t,n,B) = \frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \left(\frac{\sigma_e(x,y,z,\phi,\psi,n,B,t)}{\sigma_{max}}\right)^m \cos\phi \, d\phi \, d\psi \, dV$$
$$= \frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \left(\frac{\sigma_L(x,y,z,\phi,\psi)}{\sigma_{max}} \cdot \left(1 + \frac{t}{B}\sigma_L^2(x,y,z,\phi,\psi)\right)^{\frac{1}{n-2}}\right)^m \cos\phi \, d\phi \, d\psi \, dV \qquad [13]$$

Which is computed by the ERICA code using the multiaxial stress distribution in the specimen or component calculated by a finite element code (in this case, ANSYS).

2.1.2 Distributions Of Actual Observations

In developing the likelihood function, we need to consider actual failure events. For a given specimen with a single failure mode (flaw type) that is tested to failure, either a strength or time is observed. Thus, the events of observing a strength or a time are mutually exclusive and exhaustive. Therefore, the probability of observing any strength $(< \sigma_{max\,L})$ plus the probability of observing any time $(> 0 \text{ and } < \infty)$ is one. The term $\sigma_{max\,L}$ denotes the maximum equivalent stress in the specimen when loaded to the static test load.

The development of the likelihood needs to cover these two joint, exhaustive events. The development will be given in steps, beginning with the derivation for time distributions first.

The event of observing a time (denoted by $t < T < \infty$) occurs if and only if a time is observed for all elements of a specimen. Following the argument of Equation (1) in Batdorf and Heinisch (1978), Equations [1] and [2] of this paper can be used to express the probability of this, as (Equation [14]):

$$P(0 < T < \infty) = \exp\left\{-\frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\sigma_L(x,y,z,\phi,\psi)}{\sigma_0}\right)^m \cos\phi\, d\phi\, d\psi\, dV\right\} = \exp\left\{-IV\left(\frac{\sigma_{\max}}{\sigma_0}\right)^m\right\}. \quad [14]$$

The event of observing a time that is greater than t (denoted by $t < T < \infty$) occurs if and only if this occurs for all elements of a specimen. Arguing as for Equation [14], this can be expressed formally as (Equation [15]):

$$P(t < T < \infty) = \exp\left\{-\frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\dfrac{\sigma_L(x,y,z,\phi,\psi)}{\sigma_0}}{\left(1+\frac{t}{B}\sigma_L^2(x,y,z,\phi,\psi)\right)^{\frac{1}{n-2}}}\right)^m \cos\phi\, d\phi\, d\psi\, dV\right\} = \exp\left\{-I_t V\left(\frac{\sigma_{\max}}{\sigma_0}\right)^m\right\} \quad [15]$$

The monotonic nature of Equation [12] and hence Equation [11] implies that there can be no value of the equivalent stress less than $\sigma_L\left[1+\left(\sigma_L^2/B\right)t\right]^{1/(n-2)}$ for any element (σ_L is a function of location and angle) for which it is true that an observed time would be between t and infinity. Thus Equation [15] is valid. Since $\sigma_L\left[1+\left(\sigma_L^2/B\right)t\right]^{1/(n-2)}$ is continuous in t (for any element), Equation [15] reduces to Equation [14] when $t = 0$. Therefore Equation [14] can be employed to obtain the probability of a strength failure or that $T = 0$, as in Equation [16]:

$$P(T = 0) = 1-\exp\left\{-\frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\sigma_L(x,y,z,\phi,\psi)}{\sigma_0}\right)^m \cos\phi\, d\phi\, d\psi\, dV\right\} = 1-\exp\left\{-IV\left(\frac{\sigma_{\max}}{\sigma_0}\right)^m\right\}. \quad [16]$$

Since the events $(0 < T \le t)$ and $(t < T < \infty)$ are mutually exclusive and exhaustive of the event $(0 < T < \infty)$, the joint probability that $(0 < T \le t \text{ and a time is observed})$ is given by Equation [15] minus Equation [16] as Equation [17]:

$$P(0 < T \le t \text{ and a time is observed}) =$$
$$\exp\left\{-\frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\sigma_L(x,y,z,\phi,\psi)}{\sigma_0}\right)^m \cos\phi\, d\phi\, d\psi\, dV\right\} -$$
$$\exp\left\{-\frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\dfrac{\sigma_L(x,y,z,\phi,\psi)}{\sigma_0}}{\left(1+\frac{t}{B}\sigma_L^2(x,y,z,\phi,\psi)\right)^{\frac{1}{n-2}}}\right)^m \cos\phi\, d\phi\, d\psi\, dV\right\} \quad [17]$$
$$= \exp\left\{-IV\left(\frac{\sigma_{\max}}{\sigma_0}\right)^m\right\} - \exp\left\{-I_t V\left(\frac{\sigma_{\max}}{\sigma_0}\right)^m\right\}$$

The cumulative (marginal) distribution for an observed time of t is given by Equation [18]:

$$\begin{aligned} P(T \le t) &= P(T = 0) + P(0 < T \le t), & t > 0 \\ &= P(T = 0), & t = 0. \end{aligned} \quad [18]$$

In view of Equation [16], Equation [18] reduces to Equation [19]:

$$P(T \le t) = 1-\exp\left\{-\frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\dfrac{\sigma_L(x,y,z,\phi,\psi)}{\sigma_0}}{\left(1+\frac{t}{B}\sigma_L^2(x,y,z,\phi,\psi)\right)^{\frac{1}{n-2}}}\right)^m \cos\phi\, d\phi\, d\psi\, dV\right\} = 1-\exp\left\{-I_t V\left(\frac{\sigma_{\max}}{\sigma_0}\right)^m\right\},\ t \ge 0. \quad [19]$$

It is noted for reference, by employing equations [14] and [17], and the definition of conditional probability (or Bayes Theorem) that (Equation [20]):

$$P(T \le t \mid \text{A time is observed}) =$$
$$1-\exp\left\{\begin{aligned}&-\frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\dfrac{\sigma_L(x,y,z,\phi,\psi)}{\sigma_0}}{\left(1+\frac{t}{B}\sigma_L^2(x,y,z,\phi,\psi)\right)^{\frac{1}{n-2}}}\right)^m \cos\phi\, d\phi\, d\psi\, dV \\ &+\frac{1}{4\pi V}\iiint_V \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\sigma_L(x,y,z,\phi,\psi)}{\sigma_0}\right)^m \cos\phi\, d\phi\, d\psi\, dV\end{aligned}\right\} \quad [20]$$

Equation [20] is analogous to the distribution of "customer" observed times after a proof test has been carried out (in this case at σ_L). Equation [20] will not be used in our derivations, since it does not employ the strengths of specimens that fail on load-up.

The event of observing a strength (denoted by $\sigma_{\max e} \leq \sigma_{\max L}$, where $\sigma_{\max e}$ denotes the maximum value of σ_e in the specimen) occurs if and only if the strength for at least one element is less than the σ_L for that element. Now the events $(0 \leq \text{strength} \leq \sigma_{\max e})$ and $(\sigma_{\max e} < \text{strength} \leq \sigma_{\max L})$ are mutually exclusive and exhaustive of the event $(0 \leq \text{strength} \leq \sigma_{\max L})$. Thus, the joint probability that $(0 \leq \text{strength} \leq \sigma_{\max e} \text{ and a strength is observed})$ follows from the Batdorf element argument using Equations [1] and [2] as (Equation [21]):

$$P(0 \leq \text{strength} \leq \sigma_{\max e} \text{ and a strength is observed}) = 1 - \exp\left\{ -\frac{1}{4\pi V} \iiint_V \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \left(\frac{\sigma_e(x,y,z,\phi,\psi)}{\sigma_0} \right)^m \cos\phi \, d\phi \, d\psi \, dV \right\} = 1 - \exp\left\{ -IV \left(\frac{\sigma_{\max}}{\sigma_0} \right)^m \right\}, \quad [21]$$

Where: $\sigma_{\max e} \leq \sigma_{\max L}$.

A specimen has a censored strength observation (analogous to a time runout) if and only if the specimen strength is greater than the censored load. Clearly, the complement of Equation [21] (1 - Equation [21]) can be employed to evaluate this probability, with σ_e taking on the censored load value. Also, Equation [16] gives the probability that a strength is observed.

2.1.3 Distributions Of Censored Observations Due To Competing Failure Modes

We now consider the censoring that occurs when competing failure modes are present in a specimen. The subscript i (now) denotes the *i*th observation and the subscript j denotes the *j*th failure mode. A time is observed if and only if $\min_j\{\sigma_{ej}\} > \sigma_L$, for all elements, and thus all failure modes undergo subcritical growth. In this case, the time that is observed is the minimum of the times computed by Equation [12] for each of the failure modes. Otherwise, a critical strength for at least one failure mode is less than or equal to σ_L for at least one element, and failure will occur on initial loading.

The distribution for a specimen strength observation is derived by employing the probability of having a strength of less than or equal to x_i that is observed. Since at least one failure mode must have a critical strength less than or equal to σ_L for at least one element (in order to observe a strength upon failure), this probability is given by Equation [22]:

$$F(x_i) = 1 - S_1(x_i) S_2(x_i), \quad 0 \leq x_i \leq \sigma_{\max L}, \quad [22]$$

Where, for simplicity, the setup is given for two competing failure modes, S denotes the survivor function based on Equation [21] (the probability that the critical strength is greater than x_i, i.e., the complement of Equation [21] evaluated at x_i), and it is assumed that the failure modes act independently of each other. Differentiation of Equation [22] yields the probability density for a strength observation of x_i, which is (Equation [23]):

$$f(x_i) = f_1(x_i) S_2(x_i) + S_1(x_i) f_2(x_i). \quad [23]$$

From Equation [23] it follows that the joint probability density of a specimen failing from the first mode <u>*and*</u> at strength x_i is (Equation [24]):

$$g_1(x_i) = f_1(x_i) S_2(x_i). \quad [24]$$

For an observed strength of failure mode 1 (standard fast fracture data point or failure on loading) the probability density function f_1 is given by the derivative of Equation [21] with respect to strength, while for S_2 the complement of Equation [21] is employed, since it is a censored strength observation with respect to failure mode 1.

A similar expression for g_2 holds for mode 2. Since Equation [23] conditioned on $\{0 \leq x_j \leq \sigma_{\max L}\}$ must integrate to unity and the failure modes are independent of each other, the event of failure by one is mutually exclusive of the other. Also, both events are exhaustive. Thus, the likelihood for an observed strength and observed failure mode is given by either g_1 or g_2.

The distribution for a specimen time observation is built on the development of Equations [14], [15], [17], and [19]. Since all failure modes must have critical strengths greater than $\sigma_{\max L}$ for all elements (in order to observe a time upon failure), and the observed time is the minimum of all possible times, this probability, F, is given formally by Equation [25]:

$$\mathsf{F}(y_i) = S_1(\sigma_{\max L}) S_2(\sigma_{\max L}) - S_1(\sigma_{\max e1i}) S_2(\sigma_{\max e2i}) = S_1(\sigma_{\max L}) S_2(\sigma_{\max L}) - \mathsf{S}_1(y_i) \mathsf{S}_2(y_i), \; y_i > 0, \quad [25]$$

Where the dependency on y_i is given from Equation [11] by Equation [26]:

$$\sigma_{eji} = \sigma_L \left[1 + \frac{\sigma_L^2}{B_j} y_i \right]^{\frac{1}{n_j - 2}}, \quad [26]$$

and where the survivor function S_j for strengths is given by Equation [14] and the survivor function S_j for times is given by Equation [15] for each failure mode. The monotonic nature of Equation [26] implies that there can be no value of the equivalent stress less than the value of Equation [26] for any element such that an observed time would be greater than y_i. This also implies that a y_i censored by another failure mode, j', could not have an equivalent stress less than σ_{eji}, otherwise by the monotonicity of Equation [26] t' would have been observed. Thus, Equation [25] is valid. Differentiation of Equation [25] yields the probability density f for a time observation of y_i, which is (Equation [27]):

$$f(y_i)=f_1(y_i)S_2(y_i)+S_1(y_i)f_2(y_i) \qquad [27]$$

The first term on the right-hand side of Equation [27] covers the situation in which the observed time is due to failure mode 1, and the second term covers the situation due to failure mode 2. From Equation [27] it follows that the joint probability density of a specimen failing from the first mode *and* at time y_i is (Equation [28]):

$$h_1(y_i)=f_1(y_i)S_2(y_i). \qquad [28]$$

For a time observed for failure mode 1, the probability density function f_1 is given by the derivative of Equation [17] with respect to time, while Equation [15] (or the complement of Equation [19]) is used, with t equal to the time to failure for S_2, since it is a censored time observation with respect to failure mode 1.

A similar expression for h_2 holds for mode 2. As in the case for a strength observation, since Equation [27] conditioned on $y_i > 0$ must integrate to unity and the failure modes are independent of each other, the event of failure by one is mutually exclusive of the other, and both events are exhaustive. Thus, the likelihood for an observed time and observed failure mode is given by either h_1 or h_2.

2.1.4 The Complete Likelihood Function

The likelihood of an individual observation is defined approximately as the probability $[F(x+\Delta)-F(x)]$, where F is the cumulative distribution function and Δ is sufficiently small.

Dividing by Δ and taking the limit as Δ approaches zero gives the density function (when it exists). Since Δ is a constant scalar, the likelihood can be defined as the density evaluated at an observation. In the case of censoring or runouts, the probability of the event is used as the likelihood (see Schenk, et. al., 1998). For computational reasons, the natural logarithm of the likelihood is usually taken (Nelson, 1982).

Now, for observed failures each observation is either a failure strength or a time at which failure occurred. Thus Equation [24] multiplied by dx_i gives the probability of an observed failure strength due to failure mode 1 in the interval from x_i to $x_i + dx_i$. Likewise, Equation [28] multiplied by dy_i gives the probability of an observed time due to failure mode 1 in the interval from y_i to $y_i + dy_i$. This forms the basis for developing the likelihood function for observed failures. Let:

$$\delta_{ji} = 1, \text{ if the ith specimen fails from mode j with a strength observation}; \quad = 0, \text{ otherwise} \qquad [29]$$

$$\gamma_{ji} = 1, \text{ if the ith specimen fails from mode j with a time observation}; \quad = 0, \text{ otherwise} \qquad [30]$$

Where:

$$\delta_{1i}+\delta_{2i}+\gamma_{1i}+\gamma_{2i}=1. \qquad [31]$$

Then the likelihood for an observed strength x_i and observed failure mode, or an observed time y_i and observed failure mode is (Equation [32]) from Equations [24] and [28], the similar expressions for g_2 and h_2, and the definitions of Equations [29], [30], and [31]:

$$\begin{aligned} L_i &= [f_1(x_i)S_2(x_i)]^{\delta_{1i}}[S_1(x_i)f_2(x_i)]^{\delta_{2i}}[f_1(y_i)S_2(y_i)]^{\gamma_{1i}}[S_1(y_i)f_2(y_i)]^{\gamma_{2i}} \\ &= \left[f_1(x_i)^{\delta_{1i}}S_1(x_i)^{\delta_{2i}}f_1(y_i)^{\gamma_{1i}}S_1(y_i)^{\gamma_{2i}}\right]\cdot \\ &\quad \left[S_2(x_i)^{\delta_{1i}}f_2(x_i)^{\delta_{2i}}S_2(y_i)^{\gamma_{1i}}f_2(y_i)^{\gamma_{2i}}\right] \end{aligned} \qquad [32]$$

For future reference, note that the bracketed terms in the last line of Equation [32] each only involve failure mode 1 or failure mode 2. The log likelihood is, by definition, the logarithm of the likelihood of the complete data set, which under independence is the product of the L_i. This yields Equation [33]:

$$\begin{aligned} l = \sum_{i=1}^{n}\ln L_i &= \sum_{i=1}^{n}\ln\left[f_1(x_i)^{\delta_{1i}}S_1(x_i)^{\delta_{2i}}f_1(y_i)^{\gamma_{1i}}S_1(y_i)^{\gamma_{2i}}\right] \\ &\quad +\ln\left[S_2(x_i)^{\delta_{1i}}f_2(x_i)^{\delta_{2i}}S_2(y_i)^{\gamma_{1i}}f_2(y_i)^{\gamma_{2i}}\right] \end{aligned} \qquad [33]$$

as the log likelihood of the observed strengths or times and failure modes for the data set of actual failure observations.

Any terms accounting for censored observations due to test termination must be added to Equation [33]. These type of observations censor all failure modes and, hence, the appropriate survivor term needs to be entered in Equation [33] in each bracket. Then since the bracketed terms of the completed Equation [33] only involve a single mode, the maximum of Equation [33] is obtained by maximizing individually each of the summed bracketed terms. But, due to the nature of the δ_{ji} and γ_{ji} for the observed strengths and times (i.e., within a bracket, one and only one is unity and the others are zero), the maximization of an individual sum of brackets is obtained from a standard censored data analysis.

In order to obtain the complete log likelihood function that will also account for runout observations, let:

$$\alpha_{ji} = 1, \text{ if the ith specimen is a censored strength observation (unknown failure mode)}; \quad = 0, \text{ otherwise} \qquad [34]$$

$$\beta_{ji} = 1, \text{ if the ith specimen is a time runout (unknown failure mode)}; \quad = 0, \text{ otherwise} \qquad [35]$$

Where (Equation [36]):

$$\delta_{1i}+\delta_{2i}+\gamma_{1i}+\gamma_{2i}+\alpha_{1i}+\alpha_{2i}+\beta_{1i}+\beta_{2i}=1. \qquad [36]$$

Then the likelihood for an observed strength x_i and observed failure mode, or an observed time y_i and observed failure mode, or an observed censored strength at x_i, or an observed runout at y_i is:

$$\begin{aligned} l = \sum_{i=1}^{n}\ln L_i &= \sum_{i=1}^{n}\ln\left[f_1(x_i)^{\delta_{1i}}S_1(x_i)^{\delta_{2i}}f_1(y_i)^{\gamma_{1i}}S_1(y_i)^{\gamma_{2i}}S_1(x_i)^{\alpha_{1i}}S_1(y_i)^{\beta_{1i}}\right] \\ &\quad +\ln\left[S_2(x_i)^{\delta_{1i}}f_2(x_i)^{\delta_{2i}}S_2(y_i)^{\gamma_{1i}}f_2(y_i)^{\gamma_{2i}}S_2(x_i)^{\alpha_{2i}}S_2(y_i)^{\beta_{2i}}\right] \end{aligned} \qquad [37]$$

Where: $S_j(x_i)^{\alpha_{ji}}$ is defined by the complement of Equation [21], while $S_j(y_i)^{\beta_{ji}}$ is defined by Equation [15].

2.2 DEVELOPMENT OF CONFIDENCE BOUNDS

Methods for obtaining likelihood ratio confidence bounds of parameters for individual failure modes and reliabilities at a fixed stress and/or strengths at a fixed probability of failure are given in Cuccio, et. al., (1994). The likelihood ratio method (Cox and Oakes, 1984) is based on the direct use of the likelihood ratio statistic, Equation [38]:

$$W(\beta) = 2\left[\hat{l} - \hat{l}_\beta\right], \quad [38]$$

Where: l is the appropriate bracketed term(s) of Equation [37], $\hat{l}$ is the value of the log likelihood function evaluated at the joint maximum likelihood estimate of all parameters, β is any parameter, and $\hat{l}_\beta$ is the value of the log likelihood function evaluated at a fixed value of β and the maximum likelihood estimates of all other parameters conditional on the given value of β. $W(\beta)$ has, approximately, a chi-squared distribution with one degree of freedom. This yields a $1-\alpha$ confidence region, on the parameter β as (Equation [39]):

$$\left\{\beta : W(\beta) \le \chi^2_{1,\alpha}\right\}. \quad [39]$$

Obtaining confidence limits on a quantity such as a reliability requires substitution for, and elimination of, one of the distribution parameters, e.g., σ_0, for a particular failure mode. If only one failure mode is involved, then this can be carried out in a straightforward manner, maximizing the conditional likelihood function in terms of the introduced quantity. A much simpler approach, however, is to recognize that making the substitution is equivalent to imposing a constraint on the conditional likelihood, an observation that applies equally well to the situation of competing failure modes (cf. Cuccio, et. al., 1994). Thus, we have a constrained optimization problem comprised of the log likelihood function given in Equation [37] with a constraint equation defining the particular conditions for which the confidence limits are desired.

2.2.1 Likelihood Ratio Confidence Bounds For A Specimen Or Component Probability At A Specified Time

For a specified design time, t, the reliability of a component (again with two independent failure modes) is given by Equation [40]:

$$R(t) = S_1(t) S_2(t), \quad [40]$$

Where from Equation [19]:

$$S_j = \exp\left\{ -\frac{1}{4\pi V_c} \iiint_{V_c} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \left[\frac{\dfrac{\sigma_{Lc}(x,y,z,\phi,\psi)}{\sigma_{0_j}}}{\left(1 + \dfrac{t}{B_j}\sigma^2_{Lc}(x,y,z,\phi,\psi)\right)^{\frac{1}{n_j-2}}} \right]^{m_j} \cos\phi \, d\phi \, d\psi \, dV \right\}, \quad [41]$$

and the subscript C refers to a reference component (possibly a specimen type). Taking the logarithm of Equation [40] yields:

$$\gamma = \ln\left(\frac{1}{R(t)}\right) = \ln\left(\frac{1}{S_1(t)}\right) + \ln\left(\frac{1}{S_2(t)}\right). \quad [42]$$

Noting that Equation [42] imposes a constraint on the optimization of l, given by the complete Equation [37], $\hat{l}_\beta$ can be obtained by a constrained optimization as the solution of Equation [43]:

$$\text{Maximize } l, \text{ subject to: } \gamma = \ln\left(\frac{1}{S_1(t)}\right) + \ln\left(\frac{1}{S_2(t)}\right). \quad [43]$$

This setup produces a nice division between the specimen data through l and the component characteristics through the constraint in Equation [43]. The capability for this type of analysis is given by use of the codes ERICA and CERAMIC developed by AlliedSignal Engines. Note that the solution given by Equation [43] is employed to find the value of γ that meets the bounds of Equation [39]. γ is then inverted to obtain $R(t)$ for specified values of t.

For illustration purposes, this procedure has been applied to data on tensile specimens of a common size tested at 2200°F (1204°C) for fast fracture strengths and stress rupture lives with a stress rupture loading of 54.4 ksi (375 MPa); see Table 1 (Wu, et. al, 1995). There are two competing failure modes: internal and surface; one censored strength observation having an unidentified fracture origin; two time runouts; and 21 strength and 21 time observations.

Analyses were performed using the CERAMIC/ERICA probabilistic life prediction codes. The maximum likelihood parameter estimates for the internal failure mode are: $\hat{\sigma}_O$ = 926.5, $\hat{m}$ = 8.13, $\hat{B}$ = 10,000, and $\hat{n}$ = 25.4; while the estimates for the surface failure mode yield values of: $\hat{\sigma}_O$ = 1,817.0, $\hat{m}$ = 7.99, $\hat{B}$ = 5,655, and $\hat{n}$ = 16.2. The likelihood ratio confidence bounds that result from solving Equation [39] for selected values of t for the internal failure mode are shown in Figure 2. The curve flattens for "small" times due to the jump at $t = 0$ in the time distribution. This phenomenon generally holds in the time domain.

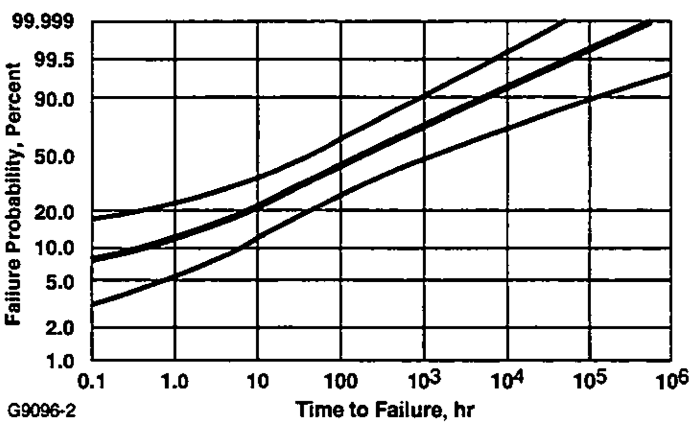

Figure 2. Resulting Weibull Plot for Specimen or Component Life Prediction Including Likelihood Ratio Confidence Bounds

Table 1. Pooled Fast Fracture and Static Fatigue Data Set From NT154 Silicon Nitride – ORNL Tensile Buttonhead Specimens (Wu, et. al., 1995)

Fast Fracture Data at 2200°F (1204°C)		Static Fatigue Data at 2200°F (1204°C) and 54.4 ksi (375 MPa)	
Failure Strength, ksi (MPa)	Failure Mode	Time to Failure, hours	Failure Mode
52.80 (364.0)	Internal	0.40	Internal
60.30 (415.8)	Internal	0.50	Internal
61.40 (423.3)	Internal	0.84	Internal
61.80 (426.1)	Internal	6.00	Surface
65.30 (450.2)	Internal	7.00	Surface
66.30 (457.1)	Internal	7.00	Surface
66.30 (457.1)	Internal	12.00	Internal
67.50 (465.4)	Internal	12.00	Internal
69.50 (479.2)	Internal	24.00	Surface
71.20 (490.9)	Internal	40.00	Internal
71.80 (495.0)	Censored	43.00	Surface
72.00 (496.4)	Surface	66.00	Internal
73.40 (506.1)	Internal	80.00	Surface
74.30 (512.3)	Internal	120.00	Surface
76.20 (525.4)	Internal	141.00	Internal
78.30 (539.9)	Internal	153.00	Internal
79.30 (546.8)	Internal	253.00	Surface
80.10 (552.3)	Internal	257.00	Surface
85.00 (586.1)	Internal	266.00	Internal
86.90 (599.2)	Internal	500.00	Runout
96.40 (664.7)	Internal	500.00	Runout

SUMMARY AND CONCLUSIONS

The methodology presented in this paper is generic enough to be applied to cyclic fatigue data analysis and life prediction as well, simply by changing the underlying stress-time transformation (Equation [12]) appropriately. The methodology is implemented into the CERAMIC/ERICA probabilistic life prediction codes, and currently allows the computation of maximum likelihood best estimates for all material parameters (Weibull modulus m, characteristic strength, slow crack growth parameters B and n), including the likelihood ratio confidence bounds of all concurrently acting (competing) flaw populations in a single run using a pooled set of fast fracture and static fatigue data.

By doing so, this approach also reveals whether slow crack growth has been the only time-dependent failure mode acting during the static fatigue experiments. Consider Figure 1, for example. The analysis of the fast fracture data alone resulted in a Weibull modulus m of 18.55. One would expect that by pooling the fast fracture data with the static fatigue data, a higher Weibull modulus should be computed, due to the increased amount of data available for parameter estimation.

Again, this is only true if the underlying assumption that slow crack growth is the only time-dependent failure mode actng on the specimens during the static fatigue experiments is correct. In this case, the pooled data analysis resulted in a lower Weibull modulus m of 13.60, and a fairly low slow crack growth exponent n of 32.5 (see Figure 3).

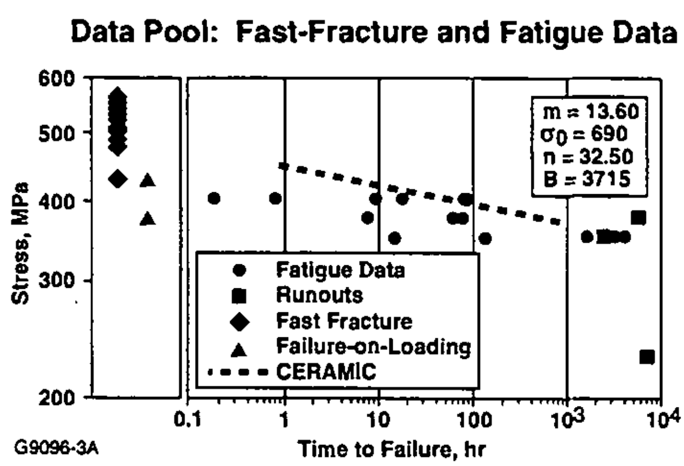

Figure 3. Pooled Data Analysis of AS-800 Silicon Nitride Fast Fracture and Static Fatigue Data (2200°F/1204°C)

On the other hand, dynamic fatigue experiments resulted in a very high n parameter value of 185. In the case of the static fatigue experiments, it could be shown that creep damage accumulation contributed to comparatively low times to failure, and therefore, to this artificially low slow crack growth exponent. Since the static fatigue data have been heavily biased by creep, larger scattering resulted for the pooled data set, which in turn is expressed by the decreased Weibull modulus.

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