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IDENTIFICATION OF MISTUNING CHARACTERISTICS OF BLADED DISKS FROM FREE RESPONSE DATA - PART II

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ABSTRACT

The focus of the present investigation is on the estimation of the dynamic properties, i.e. masses, stiffnesses, natural frequencies, mode shapes and their statistical distributions, of turbomachine blades to be used in the accurate prediction of the forced response of mistuned bladed disks. As input to this process, it is assumed that the lowest natural frequencies of the blades alone have been experimentally measured, for example in a broach block test.

Since the number of measurements is always less than the number of unknowns, this problem is indeterminate in nature. Three distinct approaches will be investigated to resolve the shortfall of data. The first one relies on the imposition of as many constraints as needed to insure a unique solution to this identification problem. Specifically, the mode shapes and modal masses of the blades are set to their design/tuned counterparts while the modal stiffnesses are varied from blade-to-blade to match the measured natural frequencies. The second approach, based on the maximum likelihood principle, yields estimates of all the structural parameters of the blades through the minimization of a specified "cost function". Finally, the third approach provides a bridge between the first two methods being based on the second but yielding a mistuning model similar to that of the first approach. The accuracy of these three techniques in predicting the forced response of mistuned bladed disks will be assessed on simple dynamic models of the blades.

INTRODUCTION

A large number of investigations have focused, in the last thirty years, on the assessment of the effects of mistuning, i.e. small blade-to-blade variations in their mechanical and/or geometrical properties, on both the free and forced responses of the entire disk. Most notably, it was shown that the free response can exhibit mode shapes which are localized to one or a few blades at the contrary of their tuned counterparts which extend to the entire disk (Wei and Pierre, 1988a). Further, it was demonstrated that the amplitude of the forced response varies greatly within a given disk from one blade to another, sometimes by a factor of 1 to 4 or larger, even when the relative fluctuations in the blade properties are as low as 1 to 2% (see Griffin and Hoosac, 1984; Kielb and Kaza, 1984; Basu and Griffin, 1986; Lin and Mignolet, 1996; and references therein). Not only have these phenomenological observations been clarified but computational approaches have been developed to quantitatively assess these effects (Wei and Pierre, 1988b; Sinha and Chen, 1989; Castanier, Ottarsson and Pierre, 1997; Yang and Griffin, 1997; Mignolet and Lin, 1996; Lin and Mignolet, 1997; Mignolet and Hu, 1998)

In most of these analyses, the variations of the blade properties were either specified or modeled as random variables with given means, standard deviations, and probability density functions, thereby generally sidestepping the issue of *characterization* of the mistuning from experimental data. This task was the focus of two recent investigations one of which (Mignolet and Lin, 1997) has relied on the availability of measurements of the forced response of consecutive blades in a series of disks, as obtained for example from strain gauges or light probes. The methodology proposed therein quite successfully recovered not only the blade structural properties but also the somewhat elusive blade-to-blade interaction terms which are known to play an important role in the analysis of mistuning. However, the measurement of the forced response of a series of bladed disks, which is required to obtain a good characterization (means, standard deviations, correlations, probability density functions,...) of the variations of the structural properties of the blades, is a time-demanding and expensive task.

To reduce this computational cost, the determination of the mistuned properties of the blades from blade alone free response (broach block) test data was investigated by Mignolet and Rivas-Guerra (1998). Specifically, it was assumed in that study that only a few of the fundamental natural frequencies of each blade were

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experimentally determined and that no mode shape measurements were performed. Then, it was recognized that the identification of the structural parameters of the blades is in fact a severely indeterminate problem that can be resolved by either specifying additional conditions or seeking the minimization/maximization of some appropriately defined cost function.

When faced with the issue of modeling mistuning, many prior investigations have only considered variations in the stiffnesses assuming the masses and, sometimes, the mode shapes of the blades equal to their tuned/design counterparts. Proceeding in this manner provides a resolution of the indetermination by the imposition of additional constraints and leads to the method termed the Random Modal Stiffness (RMS) approach by Mignolet and Rivas-Guerra (1998). An alternative strategy was also developed in that study that seeks the structural parameters of the blades to be the "most likely" set yielding exactly the measured values of the natural frequencies. Accordingly, this second identification scheme was referred to as the maximum likelihood (ML) approach. In seeking a basis to compare the RMS and ML methods, it should be noted that the estimation of the structural parameters of the blades, per say, is not as important as the prediction of the effects of the corresponding mistuning on the response of the entire disk. Motivated by the fatigue issues associated with mistuning, the reliability of the RMS and ML methods was assessed by comparing the forced responses of entire bladed disks to specific (engine order) excitations as obtained from the original ("true") structural parameters and those estimated by the RMS and ML methods. On that basis, it was found (Mignolet and Rivas-Guerra, 1998) that the ML method consistently outperformed the RMS approach and yielded estimates of the maximum amplitude of blade response on a disk that were very close to their true values. Interestingly, the lower reliability of the RMS technique was attributed in part to its inability to include fluctuations in the mass parameters of the blades.

In this light, the focus of the present investigation is first on the continued assessment of the ML identification approach and on the clarification of its features that render it superior to the RMS formulation. Next, a third method will be presented that is based on the ML technique but yields a model, as already obtained in connection with the RMS approach, that only exhibits variability in the modal stiffnesses. For the sake of completeness, a brief review of the RMS and ML approaches will first be presented that also includes an extension of the latter methodology.

IDENTIFICATION ALGORITHMS FORMULATION

Random Modal Stiffness (RMS) Approach

The first identification strategy of mistuned bladed disks to be discussed resolves the indetermination in the estimation of the structural parameters of the blades by enforcing additional constraints. Specifically, proceeding as in many prior investigations of mistuning, it is assumed that the blade variability affects only the stiffnesses so that the mass matrix of the blade structural model is taken equal to its tuned counterpart. An additional assumption is necessary if each blade is modeled as a N-degree-of-freedom system with $N > 1$. Indeed, in this case there are at least $N(N+1)/2$ stiffness elements to be evaluated but only as many as N measured natural frequencies ω_n, n=1, 2, ..., N. In the spirit of the above discussion, it is then assumed that the mode shapes of the blades are unaffected by mistuning so that the modal matrix Φ is equal to its tuned counterpart, $\Phi = \Phi_t$. If the mode shapes are normalized with respect to the tuned mass matrix, i.e. if

$$\Phi_t^T \, M_t \, \Phi_t = I_N \tag{1}$$

where I_N denotes the NxN identity matrix, it is found that the stiffness matrix of a blade is readily obtained as

$$K_{RMS} = \Phi_t^{-T} \, \Lambda \, \Phi_t^{-1} \tag{2}$$

where Λ is the diagonal matrix containing the natural frequencies, i.e.

$$\Lambda = \mathrm{diag}\left(\omega_1^2, \omega_2^2, \ldots, \omega_N^2\right). \tag{3}$$

According to the above strategy, only the modal stiffnesses of the blades are affected by mistuning and thus only these quantities vary randomly from blade-to-blade.

Constrained Maximum Likelihood (ML) Method

In the random modal stiffness strategy described in the previous section, the indetermination in the estimation of the mass and stiffness matrices is resolved by adding enough constraints (constancy of mode shapes and mass matrix) to yield a unique solution. A different approach can be undertaken in which the unknown structural parameters are chosen to optimize a certain "cost function".

To formulate such an optimization problem, note first that small fluctuations of the structural parameters of the blades do, intuitively, appear more likely to occur than large ones, or more generally that the possible variations in these parameters are not all equally "plausible". Thus, when facing several distinct sets of structural parameters yielding natural frequencies equal to their measured values, it is suggested to select the one which has the highest probability of occurrence. This "maximum likelihood" approach (see Benjamin and Cornell, 1970, for a complete presentation) requires then that

the "true" values of the masses and stiffnesses of the model of a blade are the most likely ones given the observed values of the natural frequencies of that blade

The application of this estimation strategy to the present problem was described in details in the first part of this investigation (Mignolet and Rivas-Guerra, 1998). Specifically, introduce first the vector $\underline{X}$ that contains the unknown variations in the structural parameters of the blade, that is the deviations of the elements of its stiffness and mass matrices K and M from their tuned values. Next, it is assumed that small fluctuations in masses and stiffnesses are more likely than large ones to occur and a jointly normal (Gaussian) probability density function with zero means and prescribed covariance matrix $K_{\underline{XX}}$ is selected to model the variations in blade properties. Then, according to the above maximum likelihood philosophy, the vector $\underline{X}$ should be selected to minimize the function

$$\varepsilon = \underline{X}^T \, K_{\underline{XX}}^{-1} \, \underline{X} \, . \tag{4}$$

under the constraints that the dynamic model of a blade exhibits natural frequencies equal to their measured values, ω_j, $j = 1, 2, ..., m$. Note that the number of such measurements does not necessarily equal the number of degree-of-freedom, i.e. $m \leq N$. These conditions can mathematically be expressed as

$$C_j \equiv \det\left(K - M\,\omega_j^2\right) = 0 \qquad j = 1, 2, ..., m. \tag{5}$$

In addition to the above conditions, it is assumed here that the blade model must also satisfy the following two sets of linear constraints:

$$D_j \equiv \underline{d}_j^T\,\underline{X} = e_j \qquad j = 1, 2, ..., p \tag{6}$$

and

$$\tilde{D}_j \equiv \underline{\tilde{d}}_j^T\,\underline{X} = \tilde{e}_j \qquad j = 1, 2, ..., q. \tag{7}$$

The first set of constraints is introduced to account for possible relationships existing between the elements of the mass and stiffness matrices as a consequence of the selected dynamic model of the blades and holds the same for all blades analyzed. For example, in the spring-mass system shown in Fig. 1, it can be shown that $K_{22} = -K_{12}$ $(= k_2)$, or $K_{12} + K_{22} = 0$, and $M_{12} = 0$. On this basis, the set of Eq. (6) will be referred to as the modeling constraints.

The second set of constraints, Eq. (7), differs from the first one, Eq. (6), in that the vectors $\underline{\tilde{d}}_j$ and/or the coefficients $\tilde{e}_j$ are not the same for all blades considered, as were $\underline{d}_j$ and e_j, but rather vary from blade to blade. Such a set of constraints is encountered in particular when the dynamic model of the blade is required to match not only measured natural frequencies ω_j but also observed mode shapes $\underline{\phi}_j$. Indeed, in this case, it is necessary to also enforce the conditions $\left(K - \omega_j^2\,M\right)\underline{\phi}_j = 0$ which are linear in the elements of the stiffness and mass matrices K and M and thus can be written in the form of Eq. (7). Further, the corresponding parameters $\underline{\tilde{d}}_j$ and $\tilde{e}_j$ will involve the measured values of ω_j and $\underline{\phi}_j$, and thus will vary from blade to blade.

The separation of the modeling and mode shape constraints into two separate sets, Eq. (6) and (7), may appear unnecessary at first since they are both of the same form. Note however that the existence of the *deterministic* constraints, Eq. (6), implies that the random deviations of the masses and stiffnesses from their tuned values are not linearly independent or equivalently that a subset $\underline{X}_2$ of p of these structural parameters can be expressed as a linear combination of the remaining ones stored in the vector $\underline{X}_1$. Note, however, that this linear deterministic dependence of $\underline{X}_2$ on $\underline{X}_1$ is associated (see Lutes and Sarkani, 1997) with a singular covariance matrix $K_{\underline{XX}}$ so that the definition of ε, Eq. (4), appears unclear in view of the lack of existence of the inverse $K_{\underline{XX}}^{-1}$. In fact, this situation is not unexpected since the linear deterministic dependence of $\underline{X}_2$ on $\underline{X}_1$ implies that the statistical distribution of $\underline{X}_2$ is completely dictated by its counterpart for $\underline{X}_1$. Correspondingly, the most likely vector $\underline{X}$ will be found by maximizing the likelihood of $\underline{X}_1$ alone. Equivalently, the proposed mistuning identification approach will rely on the minimization of

$$\varepsilon_1 = \underline{X}_1^T\,K_{\underline{X}_1\underline{X}_1}^{-1}\,\underline{X}_1 \tag{8}$$

where $K_{\underline{X}_1\underline{X}_1}$ is the covariance matrix of $\underline{X}_1$ and is obtained by deleting the rows and columns of $K_{\underline{XX}}$ corresponding to the elements of $\underline{X}$ that belong to the dependent subset $\underline{X}_2$.

To complete the separation of the identification problem into independent ($\underline{X}_1$) and dependent ($\underline{X}_2$) subsets of structural parameters, note that the partitioning of $\underline{X}$ into $\underline{X}_1$ and $\underline{X}_2$ can be reversed by writing

$$\underline{X} = L_1^T\,\underline{X}_1 + L_2^T\,\underline{X}_2 \tag{9}$$

where the matrices L_1 and L_2 contain only zeros and ones.

The minimization of ε_1, under the constraints given by Eq. (5)-(7), will be accomplished by the Lagrange multipliers technique. Specifically, the new function to minimize is

$$f = \underline{X}_1^T\,K_{\underline{X}_1\underline{X}_1}^{-1}\,\underline{X}_1 + \sum_{j=1}^{m}\lambda_j C_j + \sum_{j=1}^{p}\mu_j\,D_j + \sum_{j=1}^{q}\tilde{\mu}_j\,\tilde{D}_j \tag{10}$$

where m, p, and q are the number of measured natural frequencies, modeling constraints, and observed mode shapes, respectively, and λ_j, μ_j, and $\tilde{\mu}_j$ are the corresponding Lagrange multipliers associated with each set of constraints. The minimization of f for each blade will involve its partial differentiation with respect to each element of $\underline{X}_1$ and $\underline{X}_2$ and each Lagrange multiplier. Then, setting these partial derivatives to zero provides a sufficient number of equations to estimate all the elements of the vectors $\underline{X}_1$ and $\underline{X}_2$, i.e. the masses and stiffnesses of the blade considered.

When the blade structural model is a N-degree-of-freedom system, the natural frequency constraints given by Eq. (5) are nonlinear conditions, i.e. polynomials of order N, in the elements of the stiffness and mass matrices. This nonlinearity renders a direct determination of the minimum of f a difficult problem. To palliate this situation, it was proposed in the first part of this investigation (Mignolet and Rivas-Guerra, 1998) to linearize the frequency constraints with respect to the tuned values of the stiffnesses and masses. This approximation can in fact be viewed as the first step in an iterative solution of the exact minimization of f, Eq. (10). Specifically, proceeding with a first order Taylor expansion of Eq. (5) around estimates K' and M' of the stiffness and mass matrices yields (see Rivas-Guerra, 1997, for a proof)

$$1 + \mathrm{trace}\left[\left(\Delta K - \omega_j^2\,\Delta M\right)\left(K' - \omega_j^2\,M'\right)^{-1}\right] = 0 \tag{11}$$

for $j = 1, 2, ..., m$ where $\Delta K = K - K'$ and $\Delta M = M - M'$. Expanding the trace in the above equation leads to the following linearized form of this constraint as

$$C_j \equiv 1 + \underline{a}_j^T\left(\underline{X} - \underline{X}'\right) = 0 \tag{12}$$

where the vector $\underline{a}_j$ depends on the current estimates K' and M' of the stiffness and mass matrices, or equivalently $\underline{a}_j = \underline{a}_j\left(\underline{X}'\right)$.

At this point, the minimization of Eq. (10) can be performed with the linearized constraints, Eq. (12). Specifically, it is found that the deviations in structural properties stored in the vectors $\underline{X}_1$ and $\underline{X}_2$ satisfy the symmetric system of equations

$$\begin{bmatrix} 2K^{-1}_{\underline{X}_1\underline{X}_1} & [L_1\,\underline{a}] & [L_1\,\underline{d}] & [L_1\,\underline{\tilde{d}}] & 0 \\ [L_1\,\underline{a}]^T & 0 & 0 & 0 & [L_2\,\underline{a}]^T \\ [L_1\,\underline{d}]^T & 0 & 0 & 0 & [L_2\,\underline{d}]^T \\ [L_1\,\underline{\tilde{d}}]^T & 0 & 0 & 0 & [L_2\,\underline{\tilde{d}}]^T \\ 0 & [L_2\,\underline{a}] & [L_2\,\underline{d}] & [L_2\,\underline{\tilde{d}}] & 0 \end{bmatrix} \begin{bmatrix} \underline{X}_1 \\ \underline{\lambda} \\ \underline{\mu} \\ \underline{\tilde{\mu}} \\ \underline{X}_2 \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{g} \\ \underline{e} \\ \underline{\tilde{e}} \\ \underline{0} \end{bmatrix} \quad (13)$$

where $\underline{\lambda}$, $\underline{\mu}$, $\underline{\tilde{\mu}}$, $\underline{e}$, and $\underline{\tilde{e}}$ are the vector containing the values of λ_j, μ_j, $\tilde{\mu}_j$, e_j, and $\tilde{e}_j$, respectively. Further, the elements g_j of the vector $\underline{g}$ are defined as

$$g_j = -1 + \underline{a}_j^T\, L_1^T\, \underline{X}_1' + \underline{a}_j^T\, L_2^T\, \underline{X}_2' \qquad \text{for } j = 1, 2, \ldots, m. \quad (14)$$

Finally, the notation $[L_1\,\underline{a}]$ defines the matrix $\lfloor L_1\underline{a}_1 \;\; L_1\underline{a}_2 \;\; \cdots \;\; L_1\underline{a}_m \rfloor$ and similarly for $[L_1\,\underline{d}]$ and $[L_1\,\underline{\tilde{d}}]$.

It was found in general that the solution of Eq. (13) with the frequency constraints linearized with respect to the tuned system, i.e. $\underline{X}_1' = \underline{X}_2' = 0$, yield a blade model whose natural frequencies match very well their measured counterparts, ω_j, $j = 1, 2, \ldots, m$, except when two or more of these values are very close to each other. In these near veering conditions, the minimization of Eq. (10) with the nonlinear frequency constraints was achieved iteratively by solving Eq. (13) first with $\underline{X}_1' = \underline{X}_2' = 0$. Then, the corresponding solution vectors $\underline{X}_1$ and $\underline{X}_2$ were used to update the estimates K' and M' of the stiffness and mass matrices in Eq. (11), or equivalently the values of $\underline{X}_1'$ and $\underline{X}_2'$, and the process was repeated until the natural frequencies of the blade model matched their measured counterparts to an acceptable level of accuracy (see Delor, 1998, for more details).

Two important observations can be drawn from the above developments. First, the RMS approach can be viewed as a special case of the constrained ML method in which the mode shapes and the masses have been fixed to their tuned counterparts by using Eq. (7) and (6), respectively. Second, note that the ML formulation developed above is equivalent to the constrained minimization of a weighted sum of the square deviations of the structural parameters from their tuned values. Thus, given a set of natural frequencies, the maximum likelihood estimates of the structural parameters always correspond to the masses and stiffnesses closest, in the above sense, to their tuned counterparts that agree with these frequencies. Certainly, the corresponding dynamic model of the blade is the most likely one to be present but there exists a nonzero probability for larger excursions of the structural parameters from their tuned values. It is thus concluded that the ML method will in general underpredict somewhat the level of mistuning. Since the RMS formulation is a special case of the ML approach, it can be expected that it will also lead to an underprediction of the mistuning.

BLADE ALONE MODE SHAPE MISTUNING

The first part of this investigation (Mignolet and Rivas-Guerra, 1998) has focused in particular on demonstrating that the identification of the structural parameters of mistuned blades must include an estimation of both stiffness and mass variations to obtain reliable results in a subsequent forced response analysis of the corresponding mistuned bladed disks. The validity of this conclusion was supported by two separate analyses of a one-degree-of-freedom per blade model. First, a thorough parametric study was undertaken that highlighted the improved reliability of the ML approach, which estimates both stiffness and mass variations, over the RMS method which does not account for any possible mass mistuning. Second, the natural frequencies of the simplest mistuned system, i.e. the uniformly mistuned bladed disk, were determined in terms of the blade alone frequency to estimate the impact of the blade mass variability on the behavior of the response of the entire system. The goal of the present section is to perform a similar analysis to assess the effects of the variability of the blade alone mode shapes.

Importance and Prediction

Before proceeding with any mathematical development, it is desired first to assess numerically the importance of modeling the variations of the blade alone mode shapes on the reliability of the estimates of the forced response statistics of the corresponding mistuned bladed disks. To this end, the two-degree-of-freedom per blade model shown in Fig. 1 was considered and its tuned parameters were selected as those of Model 1. Mistuning was simulated, here and in all ensuing sections, by modeling the blade parameters (k_1, k_2, m_1, and m_2) as independent normal random variables, each one of which was characterized by a coefficient of variation of 0.5%, i.e. $\sigma_{m_1}/m_{1t} = \sigma_{m_2}/m_{2t} = \sigma_{k_1}/k_{1t} = \sigma_{k_2}/k_{2t} = 0.005$.

Note that the choice of this model was motivated by the finding, in the first part of this investigation, that the ML estimates of the coefficients k_1, k_2, m_1, and m_2 led to a statistical distribution of the first mode shape that closely matched its true counterpart (see Fig. 11 in Mignolet and Rivas-Guerra, 1998). Thus, the corresponding ML results can be considered representative of a successful estimation of the modal masses and stiffnesses and mode shapes of the blades. Further, the estimation of the blade parameters k_1, k_2, m_1, and m_2 in the absence of mode shape variability on the first mode was accomplished by the above ML formulation where one constraint of the form of Eq. (7) was used with $\underline{\phi}_1$ equal to its tuned counterpart, $\underline{\phi}_1^t$, for all blades.

To assess the worthiness of the constrained ML method, 240,000 blades were considered and their structural parameters were estimated from their natural frequencies by using the ML approach, its first mode constrained counterpart, and the RMS method. Then, these 240,000 blades were grouped into 10,000 mistuned disks whose responses to resonant engine order excitations of the form shown in Fig. 1 were computed. Note that the two natural frequencies of the blades were used in the ML computations and that the covariance matrix was set to its exact form. Further, these comments also hold for all ensuing ML results. Shown in Fig. 2 are the relative errors in the mean and standard deviation of the maximum amplitude of response of mass 2 found on each disk and corresponding to each of the three methods. It is clearly seen from these figures that the use of an identification methodology that does not include the variability of the mode shapes can lead to large errors in the subsequent prediction of the forced response of the corresponding mistuned bladed disks, especially at large engine orders.

Recognizing further that the difference between the RMS and mode constrained ML methods lies in the neglect, in the former approach, of the variability in the modal masses, provides a confirmation of the finding of the first part of this investigation that mass mistuning should indeed be estimated to obtain reliable forced response estimates.

Having established the importance of estimating the mode shape variability, it is necessary to clarify how and how well this information can be extracted from the sole knowledge of the fundamental natural frequencies of the blades. To this end, note first that the variation of any structural parameter will in general produce a change in both the natural frequencies and the corresponding mode shapes. Equivalently, the variations in frequencies and modes are linked to each other through the fluctuations of the structural parameters that created them. If this link is "strong" enough or, in a statistical terminology, if the coefficient of correlation between the random changes in natural frequencies and modes is large enough, the knowledge of the former will provide a good basis to estimate the latter.

An approximate expression for the coefficient of correlation between natural frequencies and mode shapes can be obtained by proceeding with perturbation techniques. Specifically, denote first by K_t, M_t, ω_j^t, and $\underline{\phi}_j^t$ the tuned stiffness and mass matrices and their corresponding natural frequencies and mass normalized mode shapes, $j = 1, 2, ..., N$. Clearly, these quantities satisfy the eigenvalue problem

$$K_t\, \underline{\phi}_j^t = \left(\omega_j^t\right)^2 M_t\, \underline{\phi}_j^t \quad \text{and} \quad \left(\underline{\phi}_j^t\right)^T M_t\, \underline{\phi}_l^t = \delta_{jl} \tag{15}$$

where δ_{jl} denotes the Kronecker symbol. Next, let the variations of these properties due to mistuning be

$$\delta K = K - K_t\,; \qquad \delta M = M - M_t\,; \qquad \delta\omega_j^2 = \omega_j^2 - \left(\omega_j^t\right)^2$$
$$\text{and} \quad \delta\underline{\phi}_j = \underline{\phi}_j - \underline{\phi}_j^t \tag{16}$$

and note that they satisfy the perturbed eigenvalue problem

$$\left(K_t + \delta K\right)\left(\underline{\phi}_j^t + \delta\underline{\phi}\right) = \left[\left(\omega_j^t\right)^2 + \delta\omega_j^2\right]\left(M_t + \delta M\right)\left(\underline{\phi}_j^t + \delta\underline{\phi}\right) \tag{17a}$$

with the normalization condition

$$\left(\underline{\phi}_j^t + \delta\underline{\phi}\right)^T M_t \left(\underline{\phi}_j^t + \delta\underline{\phi}\right) = 1. \tag{17b}$$

Then, assuming a representation of the mode shape perturbation in the form

$$\delta\underline{\phi}_j = \sum_{i=1}^{N} \alpha_{ji}\, \underline{\phi}_i \tag{18}$$

and keeping first order terms only in Eq. (17), lead to the expressions $\alpha_{jj} = 0$ and

$$\alpha_{ji}\left[\left(\omega_i^t\right)^2 - \left(\omega_j^t\right)^2\right] = \left(\underline{\phi}_i^t\right)^T\left[\left(\omega_j^t\right)^2 \delta M - \delta K\right]\left(\underline{\phi}_j^t\right) \quad i \neq j \tag{19}$$

and

$$\delta\omega_j^2 = -\left(\underline{\phi}_j^t\right)^T\left[\left(\omega_j^t\right)^2 \delta M - \delta K\right]\left(\underline{\phi}_j^t\right). \tag{20}$$

Using these relations, the statistical moments of the coefficients α_{ji} and the variations of the natural frequencies $\delta\omega_j^2$ can be determined in terms of the corresponding moments of the stiffness and mass mistuning matrices δK and δM, if they are known. In particular, in the likely case of zero mean fluctuations of the structural parameters, it is found that the coefficient of correlation between the change in natural frequency $\delta\omega_j^2$ and the mode shape variation expansion coefficient α_{ji} can be expressed as

$$corr\left(\alpha_{ji}, \delta\omega_j^2\right) = \frac{E\left[\alpha_{ji}\, \delta\omega_j^2\right]}{\sqrt{E\left[\alpha_{ji}^2\right] E\left[\left(\delta\omega_j^2\right)^2\right]}}. \tag{21}$$

Thus, following the above discussion, reliable estimates of the mode shape variability and consequently of the subsequent forced response prediction should be obtained when $corr\left(\alpha_{ji}, \delta\omega_j^2\right)$ is large. This argument is corroborated by the analysis of the two-degree-of-freedom per blade model shown in Fig. 1 (model 1) which yielded excellent mode shape (see Fig. 11 in Mignolet and Rivas-Guerra, 1998) and forced response estimates (see Fig. 2) and is associated with $corr\left(\alpha_{12}, \delta\omega_1^2\right) = 0.551$.

To further validate the adequacy of the above coefficient of correlation as an indicator of the reliability of the identification process, a second two-degree-of-freedom per blade model, defined as Model 2 on Fig. 1, was constructed that is of the same form as Model 1 but with different tuned values. For this rather arbitrary model, the coefficient of correlation was found to be $corr\left(\alpha_{12}, \delta\omega_1^2\right) = 0.116$. The small value of this quantity suggests that neither the statistical distribution of the first mode shape nor the moments of the forced response obtained by using the ML method should closely match their true counterparts. These expectations are confirmed on Fig. 3 and 4. Note, however, in both cases that the ML results are still better than their RMS counterparts which incidentally always leads to $corr\left(\alpha_{12}, \delta\omega_1^2\right) = 0$.

In practical applications, it is unclear that the statistical moments of the stiffness and mass variations would be known a priori. However, the values of these parameters obtained by application of the ML method can be used as a basis to estimate the coefficient of correlation and thus to provide a measure of confidence on the forced response prediction. In fact, performing these computations on the two models led to $corr\left(\alpha_{12}, \delta\omega_1^2\right) = 0.640$ for model 1 and 0.206 for model 2, both of which agree well with the exact values of 0.551 and 0.116.

Uniformly Mistuned Bladed Disk Results

The large differences in the prediction of the forced response obtained by using either the RMS approach or the ML method, see Fig. 2 for example, have been shown to disappear as the blade-to-blade coupling goes to zero. Further, since these two methods vary from each other by the consideration, or lack thereof, of mass and mode shape mistuning, it can be argued that the noted differences correspond to an "interaction" of the blade-to-blade coupling and the mass/mode shape mistuning terms. This observation already drawn in the first part of this investigation had led to the consideration of the simplest system that would exhibit both of these features, i.e. the uniformly mistuned system in which all the blades are identical but have stiffnesses and masses that differ from their tuned values. In the case of one-degree-of-freedom per blade, it was found that the natural frequencies of the entire disk obtained from the RMS estimates of the masses and stiffnesses differed from their true counterparts by a term that involves the product of the blade-to-blade coupling and the mass mistuning. Remarkably, it was found that the magnitude of this extra term was consistent with the errors encountered with the RMS approach and thus could be used to assess its reliability.

In this light, the focus of the present section is on the derivation of an equivalent result in the case of a multi-degree-of-freedom per blade model that exhibits mistuning in all properties including mode shape. To this end, note that the rth engine order natural frequencies of the bladed disk, $\omega_{r,j}$, and the associated modal blade deflections, $\underline{\psi}_{r,j}$, satisfy the eigenvalue problem (Thomas, 1979; Lin and Mignolet, 1997)

$$\left(K + K^{(r)}\right)\underline{\psi}_{r,j} = \omega_{r,j}^2 \left(M + M^{(r)}\right)\underline{\psi}_{r,j} \tag{22}$$

where $K^{(r)}$ and $M^{(r)}$ are NxN matrices that involve respectively the stiffness and mass coupling between blades as well as the engine order r (see Lin and Mignolet, 1997, for definitions). In the special case of a nearest-neighbors-only coupling through a stiffness matrix K_C, it is found that $K^{(r)} = 4\, K_C \sin^2\left(\frac{\pi\, r}{N}\right)$ and $M^{(r)} = 0$.

For the disk whose blades structural parameters have been estimated by the RMS approach, Eq. (22) takes the form

$$\left(K_{RMS} + K^{(r)}\right)\underline{\psi}_{r,j}^{(RMS)} = \left[\omega_{r,j}^{(RMS)}\right]^2 \left(M_t + M^{(r)}\right)\underline{\psi}_{r,j}^{(RMS)}. \tag{23}$$

At this point, recall that the dynamic model of the blades estimated by the RMS approach exhibits the correct natural frequencies so that

$$\Lambda = \Phi^T\, K\, \Phi = \Phi_t^T\, K_{RMS}\, \Phi_t \tag{24}$$

where, as in Eq. (1)-(3), Φ and Φ_t denote the blade alone modal matrices of the mistuned and tuned blades normalized with respect to their corresponding mass matrices, M and M_t, and Λ is the diagonal matrix containing the measured natural frequencies. Then, premultiplying Eq. (22) by Φ^T and Eq. (23) by Φ_t^T lead to the two relations

$$\left(\Lambda + \Phi^T\, K^{(r)}\, \Phi\right)\left(\Phi^{-1}\, \underline{\psi}_{r,j}\right) = \omega_{r,j}^2 \left(I_N + \Phi^T\, M^{(r)}\, \Phi\right)\left(\Phi^{-1}\, \underline{\psi}_{r,j}\right) \tag{25}$$

and

$$\left(\Lambda + \Phi_t^T\, K^{(r)}\, \Phi_t\right)\left(\Phi_t^{-1}\, \underline{\psi}_{r,j}^{(RMS)}\right) = \left[\omega_{r,j}^{(RMS)}\right]^2 \left(I_N + \Phi_t^T\, M^{(r)}\, \Phi_t\right)\left(\Phi_t^{-1}\, \underline{\psi}_{r,j}^{(RMS)}\right) \tag{26}$$

Introducing the mode shape mistuning matrix $\delta\Phi = \Phi - \Phi_t$, it is found that

$$\Phi^T\, K^{(r)}\, \Phi = \Phi_t^T\, K^{(r)}\, \Phi_t + \delta\Phi^T\, K^{(r)}\, \Phi_t + \Phi_t^T\, K^{(r)}\, \delta\Phi + O\left(\delta\Phi^2\right) \tag{27}$$

and similarly for $M^{(r)}$. Introducing these approximations and proceeding as in Eq. (17)-(20) yields, to first order in the mistuning,

$$\omega_{r,j}^2 = \left[\omega_{r,j}^{(RMS)}\right]^2 + 2\left(\Phi_t^{-1}\, \underline{\psi}_{r,j}^{(RMS)}\right)^T \times \Phi_t^T \left\{K^{(r)} - \left[\omega_{r,j}^{(RMS)}\right]^2 M^{(r)}\right\} \delta\Phi \left(\Phi_t^{-1}\, \underline{\psi}_{r,j}^{(RMS)}\right) \tag{28}$$

Solving this relation for the RMS based natural frequencies $\omega_{r,j}^{(RMS)}$ and expressing the RMS modal blade deflection $\underline{\psi}_{r,j}^{(RMS)}$ as a perturbation of its tuned value $\underline{\psi}_{r,j}^t$ finally yields

$$\left[\omega_{r,j}^{(RMS)}\right]^2 = \omega_{r,j}^2 - 2\left[\underline{\psi}_{r,j}^t\right]^T \left[K^{(r)} - \omega_{r,j}^2\, M^{(r)}\right]\delta\Phi\, \Phi_t^{-1}\, \underline{\psi}_{r,j}^t \tag{29}$$

As expected, it is found that the natural frequencies of the disk determined from the RMS estimates of the blade structural parameters deviate from their true values by a term that involves both the blade-to-blade coupling (in $K^{(r)}$ and $M^{(r)}$) and the mistuning in mode shape $\delta\Phi$. Note that the variations in masses are also present in Eq. (29) but do not explicitly appear because the modal matrices Φ and Φ_t have been normalized with respect to their corresponding mass matrices. In fact, in the case of a one-degree-of-freedom per blade model, one has $\Phi = \frac{1}{\sqrt{m}}$ and $\Phi_t = \frac{1}{\sqrt{m_t}}$ and Eq. (29) reduces to the relation presented in the first part of this investigation (Mignolet and Rivas-Guerra, 1998).

As before, the difference in the natural frequencies $\omega_{r,j}^{(RMS)}$ and $\omega_{r,j}$, see Eq. (29), exhibits many of the same features as the forced response of randomly mistuned bladed disks, see Fig. 2 and 4. Specifically, the presence of unmodeled mistuning in the mode shapes negatively affects the accuracy of the prediction of both the free and forced responses of the disk. Further, this effect is minimized when either the coupling is reduced or when the engine order is selected to be $r=0$ value at which the excitation emphasizes the blade alone response of the system.

Beside these important similarities in behavior, there are also some important differences which must be highlighted. For example, it can be seen that the difference between $\omega_{r,j}^2$ and $\left[\omega_{r,j}^{(RMS)}\right]^2$ vanishes for $r=0$ but the error in the predicted moments of the forced response, see Fig. 2 and 4, do not, they only admit a minimum value at this point. More importantly, Eq. (29) is indicative of a shift in natural frequency of the bladed disk which only modifies the location of the resonance, not the magnitude of the corresponding forced response. If this observation also held in connection with a randomly mistuned disk, the distribution of the maximum amplitude of blade response observed on a given disk as the excitation frequency is varied would be the same if the parameters were the true ones, or estimated by either the RMS or ML method. The results of Fig. 5 and Table 1, however, contradict this scenario and thus provide a limitation to the validity of Eq. (29) for the physical interpretation of the behavior of the RMS approach. Interestingly, note that the above results are very similar to those relating to the maximum amplitude of response that occurs on a given disk at the resonance frequency of the tuned system (see Fig. 8 in Mignolet and Rivas-Guerra, 1998) thereby demonstrating a strong consistency in the characteristics of the RMS approach.

IMPROVED RANDOM MODAL STIFFNESS (IRMS) METHOD

Introduction

The comments made in the previous section as well as the corresponding discussion conducted in the first part of this investigation (Mignolet and Rivas-Guerra, 1998) can be summarized by stating that the RMS approach provides a model of the blades that is appropriate for blade alone analyses but does not correctly represent the dynamic behavior of the blade as part of the disk. Since the prediction of the response of the entire disk is ultimately the purpose of the present identification effort, it can be questioned whether this approach can be modified to focus on the blade as a substructure of the disk system as opposed to a structure on its own.

To accomplish this task, it is suggested here to select the modal stiffnesses of the blades, keeping their modal masses and mode shapes equal to their tuned values, so that the natural frequencies of the corresponding uniformly mistuned disk in the selected engine order, not of the blades alone, be exactly matched.

An inspection of Eq. (29) immediately reveals a difficulty in this effort: since the blade structural parameters are not known, the natural frequencies of the corresponding uniformly mistuned disk cannot be predicted in advance! To palliate this situation, it is proposed here to proceed in two steps. First, the ML approach is used to produce estimates of the masses and stiffnesses of the blades from which the unknown natural frequencies of uniformly mistuned disks can themselves be estimated. Second, the modal stiffnesses of the actual blade model are then selected to exactly match these system-wide frequencies.

In this light, this new RMS-type formulation cannot be considered as an independent identification approach since it uses the results of the ML approach. Rather, it may be considered as a "model simplification" technique in which the randomness is concentrated on the modal stiffnesses of the blades. The benefits of investigating such a strategy are twofold. First, the analysis will provide additional information on the appropriateness of a modal-stiffness-only modeling of mistuning as well as on the validity of Eq. (29) to clarify the issues associated with this modeling. Second, this technique could be used to adapt the ML methodology to situations that privilege the RMS approach.

Formulation

According to the above strategy, the general formulation of the proposed technique, termed the Improved Random Modal Stiffness (IRMS) method, proceeds as follows. First, estimates of the blade stiffness and mass matrices K and M are obtained by the ML method. Then, a tuned disk is considered in which each blade is characterized by the identified stiffness and mass matrices K and M, respectively. This system possesses N natural frequencies that are associated with the rth engine order, i.e. $\omega_{r,j}$, where $j = 1, 2, ..., N$. From Eq. (22), these frequencies are solutions of the characteristic equation

$$\det\left[\left(K + K^{(r)}\right) - \omega_{r,j}^2 \left(M + M^{(r)}\right)\right] = 0 \quad j = 1, ..., N; \text{ and } r \text{ fixed} \quad (30)$$

To meet the requirements and format of the IRMS method, the corresponding identified stiffness and mass matrices K^{IRMS} and M^{IRMS}, respectively, should satisfy the following conditions:

(1) exhibit mode shapes and modal masses equal to their tuned counterparts. As in Eq.(1)-(3), this condition is equivalent to the relations

$$M^{IRMS} = \Phi_t^{-T} \Phi_t^{-1} \quad \text{and} \quad K^{IRMS} = \Phi_t^{-T} \Lambda^{IRMS} \Phi_t^{-1} \quad (31)$$

where Λ^{IRMS} denotes the diagonal modal stiffness matrix for the mistuned blade, i.e.

$$\Lambda^{IRMS} = \mathrm{diag}\left(\Lambda_{11}, \Lambda_{22}, ..., \Lambda_{NN}\right) \quad (32)$$

(2) yield system natural frequencies corresponding to the engine order r that exactly match those predicted on the basis of the ML estimates of the stiffness and mass matrices. This last condition implies that the natural frequencies $\omega_{r,j}$, $j = 1, 2, ..., N$, determined from Eq. (30) must also satisfy the relation

$$\det\left[\left(K^{IRMS} + K^{(r)}\right) - \omega_{r,j}^2 \left(M^{IRMS} + M^{(r)}\right)\right] = 0 \quad (33)$$

Introducing Eq. (31) into Eq. (33) leads to the conditions

$$\det\left[\Lambda^{IRMS} + A_j^{(r)}\right] = 0 \qquad j = 1, 2, ..., N; \quad r \text{ fixed} \quad (34)$$

where

$$A_j^{(r)} = \Phi_t^T \left[K^{(r)} - \omega_{r,j}^2 M^{(r)}\right] \Phi_t - \omega_{r,j}^2 I_N. \quad (35)$$

Once the N equations (34) have been solved for the N elements Λ_{jj} of the diagonal matrix Λ^{IRMS}, the IRMS estimates of the mass and stiffness matrices of the blade can be computed by Eq. (31). In connection with a one-degree-of-freedom per blade model, these computations are readily accomplished and yield $m^{IRMS} = m_t$ and

$$k^{IRMS} = m_t \Lambda_{11} = m_t \omega_r^2 - \left[k^{(r)} - \omega_r^2 m^{(r)}\right] \quad (36)$$

Solving Eq. (30) for the natural frequencies ω_r yields $\omega_r^2 = \left(k + k^{(r)}\right) / \left(m + m^{(r)}\right)$ so that

$$k^{IRMS} = \frac{\left(m_t + m^{(r)}\right)k + \left(m_t - m\right)k^{(r)}}{\left(m + m^{(r)}\right)}. \quad (37)$$

For the system shown in Fig. 6, the above expression simplifies further to

$$k^{IRMS} = \frac{m_t}{m}k + \frac{\left(m_t - m\right)}{m}\left[4\,k_C \sin^2\left(\frac{\pi r}{N_b}\right)\right]. \quad (38)$$

In the case of a two-degree-of-freedom system, Eq. (34) still admits a closed form solution as (Delor, 1998)

$$\Lambda_{11} = -\frac{1}{2}\left(A_{2,11}^{(r)} + A_{1,11}^{(r)}\right) \pm \frac{1}{2}\sqrt{\left(A_{2,11}^{(r)} - A_{1,11}^{(r)}\right)^2 - 4\left(\frac{A_{2,11}^{(r)} - A_{1,11}^{(r)}}{A_{2,22}^{(r)} - A_{1,22}^{(r)}}\right)\left(A_{1,12}^{(r)}\right)^2} \quad (39)$$

and

$$\Lambda_{22} = \frac{A_{2,11}^{(r)}\,A_{2,22}^{(r)} - A_{1,11}^{(r)}\,A_{1,22}^{(r)} + \left(A_{2,22}^{(r)} - A_{1,22}^{(r)}\right)\Lambda_{11}}{A_{1,11}^{(r)} - A_{2,11}^{(r)}} \quad (40)$$

from which the masses and stiffnesses of the blade can be evaluated. It is important to note from Eq. (39) that the existence of the parameter Λ_{11} is not guaranteed since the term inside the square root is not always positive. Even when it is, the corresponding solutions Λ_{11} and Λ_{22} may not greater than zero which is required to obtain a positive definite stiffness matrix K^{IRMS}. Note also that when a bonafide solution does exist, it may not be unique because of the ± sign in Eq. (39). In most cases, however, the "correct" solution exists and is quite evident; it corresponds to values $\Lambda_{jj} \approx \omega_j^2$ as can be expected by comparing Eq. (2) and (31). In the general case $N > 2$, a closed form solution for the parameters Λ_{jj} was not found but an iterative numerical technique based on Newton's method was successfully employed (see Delor, 1998, for details).

It should finally be noted that the IRMS technique described above reduces exactly to the RMS approach when r=0 since the corresponding matrices $K^{(0)}$ and $M^{(0)}$ vanish identically. This result is consistent with the motivation behind the IRMS developments since a zeroth engine order excitation favors the blade alone behavior on which the RMS method is based.

Numerical Results

The reliability of the IRMS strategy was first assessed by considering the one-degree-of-freedom per blade model shown in Fig. 6. Shown in Fig. 7 are the relative errors in the mean and standard deviation of the corresponding maximum amplitude of resonant response on the disk as a function of the engine order r for the ML, RMS, and IRMS approaches. Clearly, the IRMS method leads to excellent estimates of the forced response statistics, its errors are typically much smaller than the ones corresponding to the RMS approach (except of course for r=0 where they are equal) and of the same order of magnitude as their ML counterparts.

The results of a similar study conducted on the basis of the two-degree-of-freedom per blade models shown in Fig. 1 (models 1 and 2) can be seen on Fig. 2 and 3. Clearly, in both of these cases, the errors associated with the proposed IRMS method are typically close to their ML counterparts, except for r=0, and smaller than those corresponding to the RMS approach. Interestingly, it can be observed that the IRMS technique sometimes leads to an overprediction of the mean and standard deviation of the response which is in contradiction with the ML and RMS methods that consistently underpredict the values of these moments.

IDENTIFICATION OF MISTUNING IN THE PRESENCE OF VEERING

Small changes in the values of the masses and stiffnesses of a dynamic system usually imply correspondingly small changes in its mode shapes. However, when two or more of its natural frequencies are close together, small perturbations of the system parameters can often lead to large variations of the corresponding mode shapes. This situation, referred to as mode veering, is in particular encountered in connection with weakly coupled bladed disks where a small mistuning can produce a dramatic localization of the mode shapes (Wei and Pierre, 1988a). Note however that the veering phenomenon is not limited to the disk modes, it can also be encountered in connection with their blade alone counterparts. In fact, a recent investigation (Griffin, 1997) has demonstrated that some wide chord fan blades exhibit a series of very closely spaced natural frequencies and that the corresponding blade alone mode shapes vary dramatically from blade to blade as a consequence of the small geometric and material differences between them.

In this context, it is desired here to assess the reliability of the RMS, ML, and IRMS identification techniques when two of the blade natural frequencies are close together. To this end, the two-degree-of-freedom shown in Fig. 1 (model 2) was selected and the tuned values of the parameters k_2 and m_2 were varied to simulate different configurations. Of particular importance in the analysis of the veering phenomenon is the subsystem coupling which is here represented by the stiffness k_2. The reliability of the three identification methods will thus be assessed for different values of this parameter. In order to provide a worst case scenario, the mass m_2 was chosen, for a given stiffness k_2, so that the two natural frequencies of the system are as close as possible, i.e.

$$m_2 = \frac{k_2}{k_1 - k_2}m_1 \quad (41)$$

Shown in Table 2 are the four sets of values of k_2 and m_2 that were considered as well as the corresponding relative differences in the frequencies ω_1 and ω_2, i.e. $2\left(\omega_2 - \omega_1\right)/\left(\omega_2 + \omega_1\right)$. For each of these four cases, the reliability of the forced response estimates was evaluated for different engine order excitations and it was found that the largest errors in the mean and standard deviation occurred at larger engine orders, r = 6 to 9, only for case 4, see Fig. 4, while they were associated with r = 0 for the three other blade models and for both the RMS and ML approaches. This observation which is somewhat contrary to prior findings, see Fig. 2 and 4 for example, may be explained by noting that the natural frequencies of the corresponding tuned disks exhibit a veering at r=0. Then, relying on past findings (see for example Kruse and Pierre, 1997, and Kaiser et al., 1994), it may be argued that the amplification of the forced response by mistuning would be particularly significant for such an excitation thereby increasing the likelihood of larger errors.

Consequently, shown in Fig. 8 are the relative errors in the mean and standard deviation of the maximum amplitude of forced response of mass 2 on the disk when subjected to a zeroth order excitation. Most notable on this figure are the sharp increase in the RMS errors as the two blade alone frequencies approach each other and the much less serious worsening of the ML forced response prediction, the latter of which is consistent with the continuous decrease of the mode shape - natural frequency coefficient of correlation shown in Table 2. The improved behavior of the ML method over the RMS approach, although less dramatic, was also observed for other values of the engine order (see Delor, 1998). Note finally that the iterative process described in connection with the satisfaction of the nonlinear frequency constraints in the ML formulation was found to be required in cases 1-3 as can be expected since an increased closeness of the two frequencies implies a more limited validity of the linearized approximation given by Eq. (11)-(12).

The reliability of the IRMS method is not adequately described by Fig. 8 since this approach automatically reduces to the RMS technique for $r=0$. For other values of the engine order, it was typically found that the IRMS approach yielded errors of the same order of magnitude as the ML method for cases 3 and 4 but exhibited substantially larger errors for case 2 and, especially, case 1. In conjunction with this observation, it was also found for these last two systems that Eq. (39) and (40) did not yield acceptable values of Λ_{11} and Λ_{22}, i.e. real and positive, for a fraction of the blades analyzed that increased from 0 for $r=0$ to almost 50% for case 1 with $r=12$. This absence of an IRMS model for certain sets of measured natural frequencies was proved (Delor, 1998) to be indeed associated with the closeness of the two natural frequencies and is by itself indicative of the breakdown of the logic associated with the formulation of the IRMS strategy. In this light, the appearance of large errors in the forced response prediction for the blades for which a model was in fact obtained is not surprising. On this basis, it is tentatively suggested that the IRMS method not be used when two natural frequencies are within 5 to 10% or less of each other.

SUMMARY

This two-part investigation has focused on the estimation of the blade-to-blade variations of their structural properties, in particular masses and stiffnesses, from measurements of the lowest natural frequencies of these blades only. Practically, this identification task does not represent an end result but rather a necessary first step in the accurate prediction of the forced response of turbomachine disks that would support the blades tested. It is thus in terms of forced response accuracy that the reliability of the suggested identification strategies has been assessed.

In this second part of the investigation, the Maximum Likelihood (ML) approach introduced in Part I was reconsidered and its formulation was extended to include two sets of linear constraints to be satisfied by the estimated blade model. The first type of constraint was shown to be associated with fixed relationships between the structural parameters of the blade that are associated with the assumed dynamic model. In this light, these conditions were referred to as modeling constraints. The second type of constraints is characterized by coefficients that vary from blade to blade and is encountered in particular when requiring that the blade model exhibit a measured or specified mode shape. As in Part I, it was shown that the determination of the structural parameters according to the ML method with the constraints could generally be accomplished non-iteratively by solving one linear system of equations, although a few iterations are recommended when the natural frequencies of the blades alone are close together.

The availability of the above constrained maximum likelihood formulation enabled the assessment of the importance of mode shape mistuning on the forced response of bladed disks, a seldom discussed issue. In fact, several comparisons were made between the reliability of the forced response predictions corresponding to identified blade models in which only the modal stiffnesses vary from blade to blade (the RMS approach of Part I), both modal stiffnesses and masses fluctuate (the ML approach with mode constraints), and modes shapes as well as modal parameters are mistuned (original ML approach). The results of this analysis indicate that the inclusion and accurate estimation of the blade-to-blade variability of their mode shapes are essential to accurately predict the forced response of mistuned disks from measurements of the corresponding blade alone natural frequencies. In this context, it was demonstrated that the ML method provides reasonable to excellent estimates of the required mode shape variability because of the existence of a statistical correlation between the blade alone mode shapes and natural frequencies. Further, an approximate expression for the corresponding coefficient of correlation was derived that was shown to relate directly to the reliability of the ML forced response prediction. Interestingly, it was shown that this coefficient could also be estimated from the population of identified blade models. Practically speaking, it is then possible to gauge directly, without any prior knowledge of the mistuning actually present, the reliability of the forced response statistics produced from the measured blade alone natural frequencies.

It was demonstrated in the first part of this investigation, on the basis of a one-degree-of-freedom per blade model, that the relatively poor reliability of the random modal stiffness (RMS) technique could be justified by investigating the natural frequencies of uniformly mistuned bladed disks, not of the blades alone. A similar discussion was reconducted here on the basis of a general multi-degree-of-freedom per blade model. This generalization demonstrated that given the mass and mode shape mistuning of the blades, their modal stiffnesses can be selected to provide either accurate estimates of the blade alone natural frequencies or of the natural frequencies of the entire bladed disk, but not of both. Thus, by matching precisely the measured values of the blade alone frequencies, the RMS approach does not predict properly the behavior of the blades as components of the disk. This important observation suggested the introduction of a third technique, termed the improved random modal stiffness (IRMS) method, in which the random modal stiffnesses are in fact selected to match the behavior of the blade as part of a tuned assembly of similar blades. This approach is not per say an identification technique since it requires the prior determination of the ML estimates but it can be construed as a model simplification strategy or as a bridge between the ML and RMS formulations. As expected, the IRMS formulation led to errors in the forced response prediction that were smaller than their RMS counterparts although both of these approaches are similar in that they do not include mistuning in either mass or mode shape.

The last focus of the present investigation was on the assessment of the reliability of the three, ML, RMS, and IRMS, approaches when some of the blade alone frequencies are close together. This numerical study demonstrated an increase of the errors of the ensuing forced response prediction for all three methods but confirmed again the superiority of the ML formulation over its RMS counterpart.

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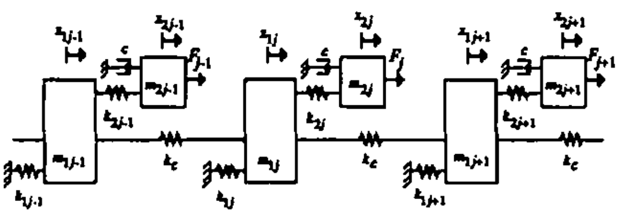
DOF	Original	RMS	ML	Error RMS	Error ML
1	0.1466	0.0443	0.1427	-69.77 %	-2.64 %
2	0.1476	0.0589	0.1476	-60.11 %	0.029 %

Table 1. Probabilities that the maximum amplitude exceeds its mean value by one standard deviation, ML and RMS approaches, two-degree-of-freedom blade model 1 excited in the first frequency band in a frequency sweep around the 9th engine order tuned frequency, $\omega \in [0.95\,\omega'_{9,1}, 1.05\,\omega'_{9,1}]$

Case	k_2 (N/m)	m_2 (kg)	Rel. Freq. Diff.	$corr(\alpha_{12}, \delta\omega_1^2)$
1	275.67	$7.313\ 10^{-6}$	2.54 %	0.1165
2	1,132.29	$3.010\ 10^{-5}$	5.14 %	0.0496
3	4,789.78	$1.284\ 10^{-4}$	10.58 %	0.0249
4	36,411.44	$1.055\ 10^{-3}$	29.74 %	0.0121

Table 2. Blade parameters and modal characteristics for the four cases considered in the veering analysis
Rel. Freq. Diff. = $2(\omega_2 - \omega_1)/(\omega_2 + \omega_1)$

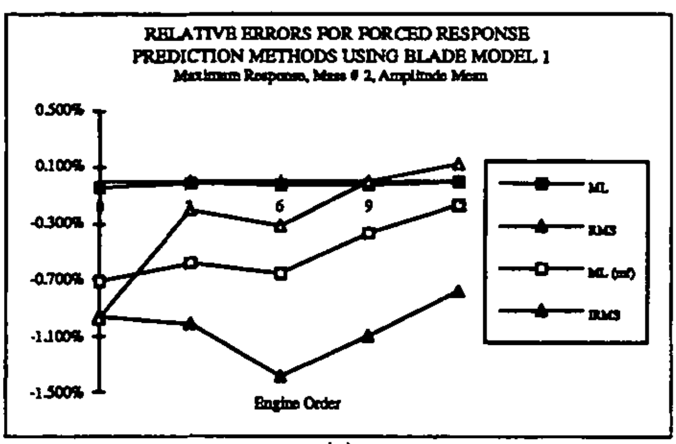


$$F_j = F_o \cos\left(\omega_r t + \frac{2\pi r(j-1)}{N_b}\right) \qquad N_b = 24 \text{ blades} \quad F_o = 1\text{ N}$$

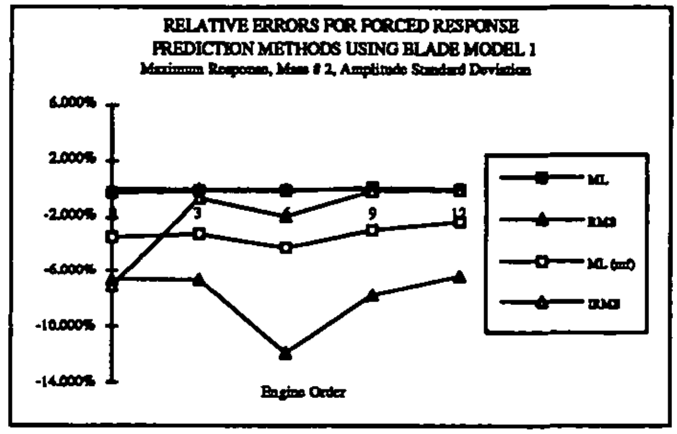
Model 1: $m_{1t} = 0.0780$ kg $m_{2t} = 0.0114$ kg $k_{1t} = 7{,}166{,}028$ N/m
$k_{2t} = 415{,}557$ N/m $c = 1.443$ Ns/m $k_c = 5{,}236{,}000$ N/m

Model 2: $m_{1t} = 0.0114$ kg $m_{2t} = 0.001055$ kg $k_{1t} = 430{,}000$ N/m
$k_{2t} = 36{,}411.44$ N/m $\zeta_1 = \zeta_2 = 1\%$ $k_c = 45{,}430$N/m
$\sigma_{m_1}/m_{1t} = \sigma_{m_2}/m_{2t} = \sigma_{k_1}/k_{1t} = \sigma_{k_2}/k_{2t} = 0.005.$

Fig. 1 Two-degree-of-freedom per blade disk models



(a)



(b)

Fig. 2 Errors (a) in the mean and (b) in the standard deviation of the maximum response of the mass m_2 as functions of the engine order. ML, ML with first mode fixed (mf), RMS and IRMS strategies, two-degree-of-freedom blade model 1 excited in the first frequency band.

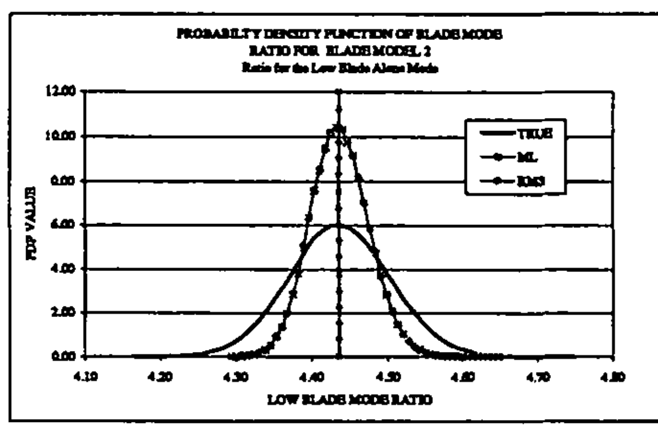
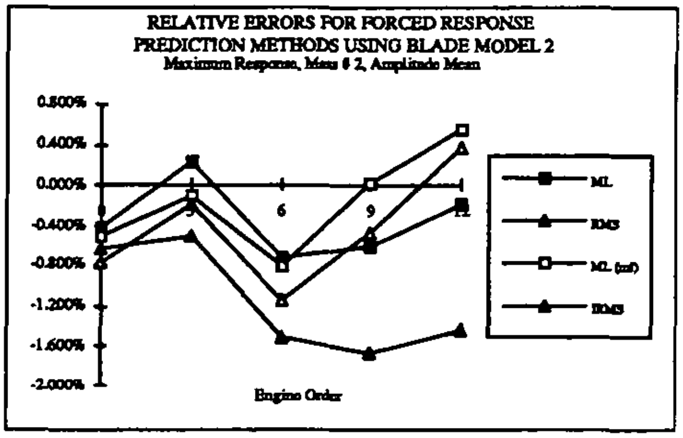
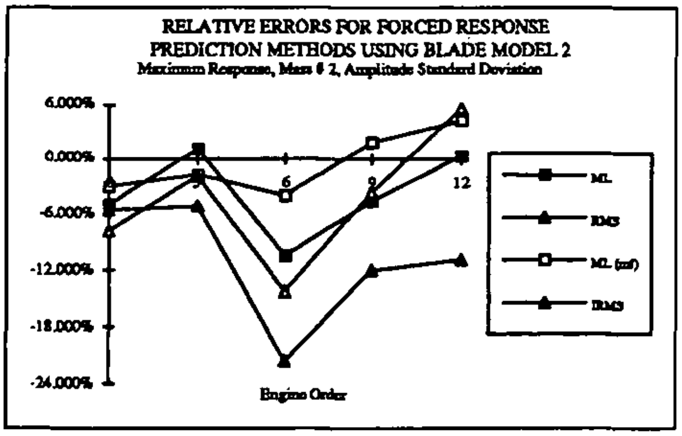


Fig. 3 Probability density function of the ratio of the modal displacements of masses 1 and 2, low frequency mode. Original blade parameters, estimated by the ML approach and estimated by the RMS technique, two-degree-of-freedom blade model 2.



(a)



(b)

Fig. 4 Errors (a) in the mean and (b) in the standard deviation of the maximum response of the mass m_2 as functions of the engine order. ML, ML with first mode fixed (mf), RMS and IRMS strategies, two-degree-of-freedom blade model 2 excited in the first frequency band.

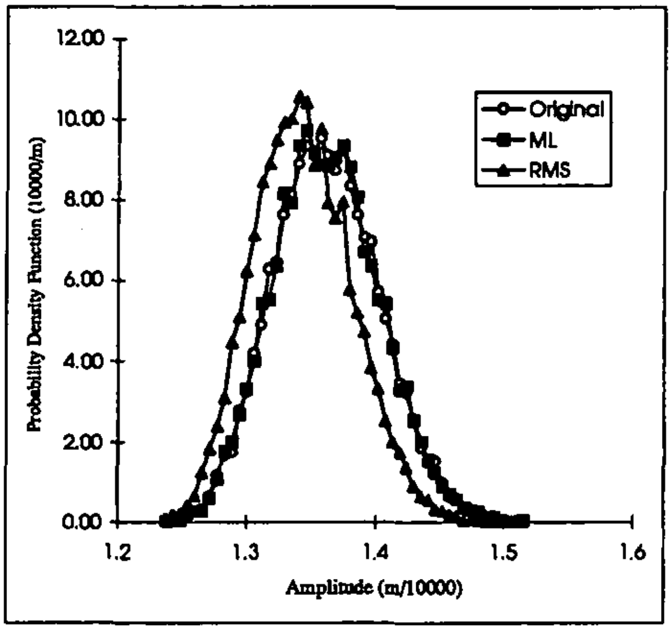
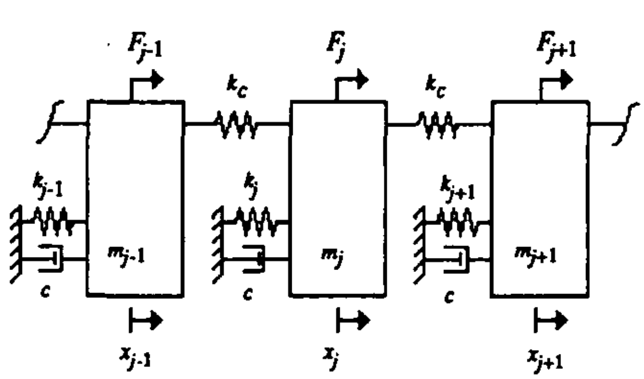


Fig. 5 Probability density function of the maximum amplitude of response of the mass m_2 for a 9th engine order excitation over a frequency sweep $\omega \in \left[0.95\,\omega^{t}_{9,1}, 1.05\,\omega^{t}_{9,1}\right]$. Original parameters, estimated by ML and RMS approaches, two-degree-of-freedom blade model 1 excited in the first frequency band.



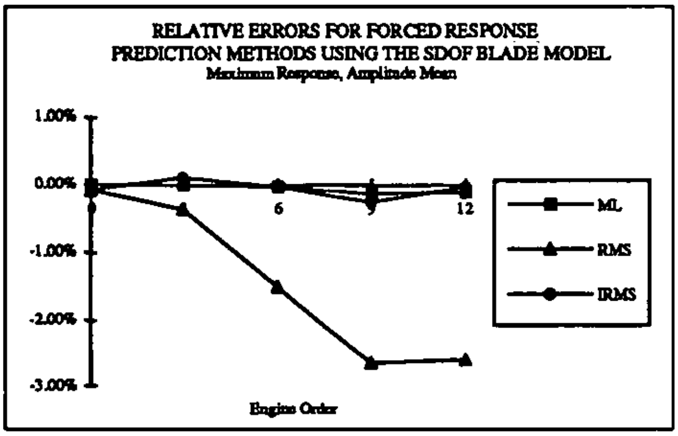
$$F_j = F_o \cos\left(\omega_r t + \frac{2\pi r(j-1)}{N_b}\right)$$

m_t=0.0114 kg k_t=430,000 N/m c=1.443 Ns/m

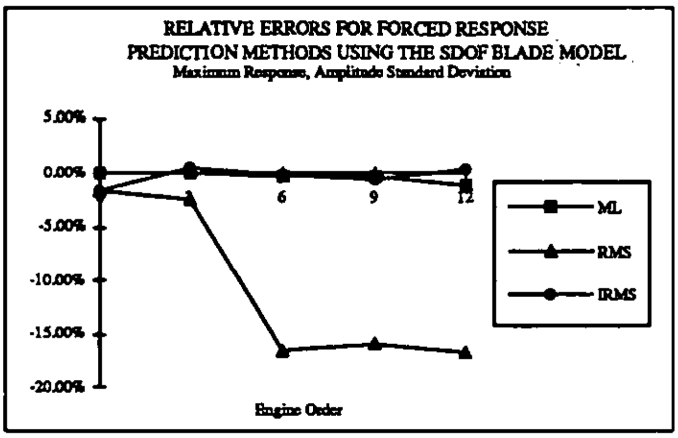
k_c=45,430 N/m N_b=24 blades $F_o = 1\,\mathrm{N}$

$$\sigma_m / m_t = 0.01\sqrt{2} \quad \sigma_k / k_t = 0.01$$

Fig.6 Single-degree-of-freedom per blade disk model.

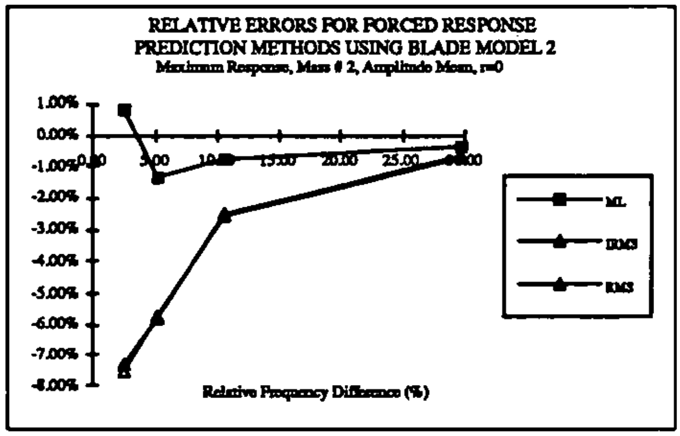


(a)

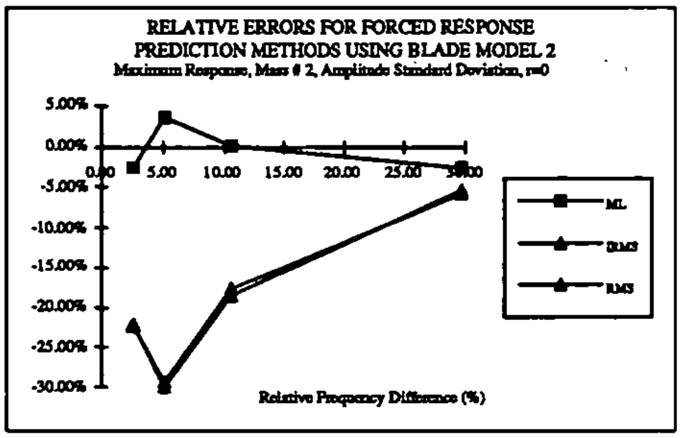


(b)

Fig. 7 Errors (a) in the mean and (b) in the standard deviation of the maximum amplitude of resonant response of the blade as functions of the engine order (*r*), ML, RMS, and IRMS approaches, single-degree-of-freedom blade model.



(a)



(b)

Fig. 8 Errors (a) in the mean and (b) in the standard deviation of the maximum response of the mass m_2 as functions of the relative difference of frequencies. ML, RMS and IRMS strategies, two-degree-of-freedom blade model 2 excited in the first frequency band