Probabilistic Structural Analysis for Advanced Space Propulsion Systems

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ABSTRACT
The paper reports on recent extensions to ongoing research into probabilistic structural analysis modeling of advanced space propulsion system hardware. The advances concern probabilistic dynamic loading, and probabilistic nonlinear material behavior.

In both cases, the reported work represents a significant advance in the state-of-the-art for these topics. Random, or probabilistic loading is normally concerned with the loading described in power spectral density (PSD) terms. The current work describes a method for incorporating random PSD's along with random material properties, damping, and structural geometry.

The probabilistic material response is concerned with the prediction of nonlinear stress-strain behavior for physical processes that can be linked to the original microstructure of the material. Such variables as grain size and orientation, grain boundary strength, etc. are treated as random, initial variables in generating stochastic stress-strain curves. The methodology is demonstrated for a creep simulation.

INTRODUCTION
Design of high performance structures such as gas turbine engine components must take increasing cognizance of realistic variations in design variables resulting from materials, manufacturing, and operations. Reliability of high performance structures must be improved over that of older designs while, at the same time, the design margins are being reduced for performance.

The current paper focuses on two areas of design analyses for turbine and turbopump applications for which probabilistic design issues are seen to be of increasing importance. The first is the area of dynamic response due to imposed dynamic loads; the second is the effect of material behavior on nonlinear structural response. Both of these topics have direct bearings on the reliability of components in turbine engine hot sections, as well as other components. In each case, the paper will report on state-of-the-art extensions to account for advanced probabilistic design concerns.

The work reported herein is part of a long-term research contract sponsored by NASA (Lewis Research Center), NAS3-24389, under the technical direction of Dr. C.C. Chamis. The goal of the project is to develop general purpose analysis procedures to permit the calculation of structural reliability for advanced space propulsion systems. The first phase of that effort has been the generation of analysis tools that will predict the uncertainty in structural response for uncertainties in geometry, material properties, and loading. Future work will integrate these procedures into generalized reliability methods.

PROBABILISTIC STRUCTURAL ANALYSIS METHODS (PSAM)

Current Issues
Significant progress already made in the development of the PSAM technology for structural modeling. The following sections extend this earlier work to include random-random loading, and random thermoviscoplastic behavior. The approach previously developed for linear structural analysis of random structures is being extended for these two new problems. The current report is limited to the finite element implementation of the software system NESSUS - Nonlinear Evaluation of Stochastic Structures Under Stress.

Approach
The NESSUS code is a very general software system organized in a modular fashion as shown in the figure below. The elements of the code include an expert system-based user interface NESSUS/Pre, a finite element analysis code adapted to probabilistic modeling NESSUS/FEM, a special database, and a probabilistic evaluation code NESSUS/FPI.

The approach that has been developed for PSAM is based on the use of perturbations of the random variables to determine structural sensitivity to each of these variables. If the sensitivity to each random variable can be estimated numerically (by FEM or other methods, including experiments), and if the statistics of the random variables are known, then the probabilistic distribution of the solution state can be computed. The computation

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makes use of the theory of joint probabilities for multivariate problems developed by Wu (1987), called the Fast Probability Integration Algorithm (FPI).

The NESSUS code is written to facilitate the generation of the perturbation data by modifying the input data in a manner consistent for each random variable. The random variable may be a point value (e.g., boundary condition) or a field variable (e.g., thickness, modulus), depending on the level of modeling accuracy that is required. Random field variables are converted into sets of independent random variables for the perturbation analysis by NESSUS/FEM.

Once the perturbations are individually analyzed, the response variable (stress or deflection at a point, natural frequency, or buckling load) is fitted to a hyper-plane or quadratic surface. The NESSUS/FPI algorithm accesses the perturbation data in the database, as well as the statistical data on each random variable. The entire distribution of the solution variable is then estimated from one set of perturbations.

The perturbations are only accurate over some distance from the central state for the perturbations. To obtain accuracy in the tails of the distribution, the NESSUS/FEM solution is reformed using the estimated set of random variables selected by FPI for that probability level. The solution is then perturbed near the new solution state in order for FPI to compute accurate distributions at the selected level of probability. The number of resolutions is generally very small (say 2-4) for realistic problems requiring solutions into the extreme tails (e.g., 0.0001 or 0.9999 cumulative probability). Comparison of this approach to Monte Carlo was shown in the article by Cruse et al. to be highly efficient and accurate.

Applications

As an example of the NESSUS capability for linear dynamic analysis, a turbine blade FEM mesh is shown in the figure below. The blade is assumed to be rigidly clamped at the transition plane, above the attachment flitchord. Three sets of random variables are included in the analysis. First, the blade is fabricated from a single crystal material with uncertain crystallographic orientations. Second, the elastic constants are also known to have variability. Finally, the machining of the turbine blade attachment results in a variable blade tilt and rotation.

FIG. 1 MODULAR CONSTRUCTION OF THE NESSUS CODE FOR PSAM

The perturbation approach is illustrated in the next figure, showing an example of a one-variable uncertainty on frequency response. The figure shows a material property to be represented by a cumulative distribution function (CDF). The figure also illustrates the notion that the dynamic response of a system changes as the system variable changes. The change is represented in PSAM by the slope of the response surface, as estimated by a perturbation analysis. If the material parameter distribution is known, and the effect of changes in the uncertain property are estimated, then the Fast Probability Integration (FPI) algorithm of Wu computes the cumulative distribution of the resulting system natural frequency, or other performance measure. The approach can treat any number of random variables, and is accurate even in the tails of the distribution of the response variable (Cruse et al., 1988).

FIG. 2 PERTURBATION DATA SUPPORTS FPI CALCULATION OF DISTRIBUTION

FIG. 3 FEM MODEL OF TURBINE BLADE FOR NESSUS MODELING
The next figure presents the computed cumulative distribution of natural frequencies of the blade for the full set of random variables. The line labeled MVFO (Mean Value First Order) represents the estimate made by a linear perturbation about the mean value state. Point 1 represents the natural frequency for about 0.003% cumulative probability. FPI defined the set of random variables that are associated with this probability level; by reforming the stiffness matrix with these values Point 2 is obtained.

![Figure 4: Cumulative Distribution for Blade Natural Frequencies](image)

In the NESSUS code, the dynamic response of a structure with stochastic variation in physical properties of the structure or loading can be synthesized in the frequency domain via modal analyses or in the time domain via direct time integration of the system equations of motion. A brief description of the analysis approaches will now be given along with basic equations to illustrate the level of stochastic variation.

**Frequency Domain Modeling**

Three levels of frequency domain modeling exist in the NESSUS code as indicated by the level content given in the table below. A brief description of each of the modeling levels will now be given with an outline of the corresponding dynamic equations of motion to explicitly demonstrate the differences in the various modeling levels. Terms such as deterministic-random, and random-random may appear to be confusing in the context of normal random structural dynamics. Level 0 is classical stochastic structural dynamics. Level 1 concerns random geometry, damping, etc., but uses scale factors for the entire structure and PSD level. Level 2 allows the random variables to be distributed within the structure and within the PSD.

**Table 1: Dynamic Analysis Modeling—Frequency Domain**

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>LEVEL 0</th>
<th>LEVEL 1</th>
<th>LEVEL 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry</td>
<td>Deterministic Structure</td>
<td>Global Uncertainty</td>
<td>Full Uncertainty</td>
</tr>
<tr>
<td>Shapes</td>
<td>Modal Analysis</td>
<td>Scaled Eigenvectors</td>
<td>Perturbed Eigenvectors</td>
</tr>
<tr>
<td>Frequency</td>
<td>Periodic</td>
<td>Uncertain Periodic</td>
<td>Random Periodic</td>
</tr>
<tr>
<td>PSD</td>
<td>Deterministic Random</td>
<td>Uncertain Random</td>
<td>Uncertain Random</td>
</tr>
</tbody>
</table>

**Level 0 - Deterministic**

In Level 0 analysis, the structure is considered to be deterministic as well as the loading, except for the case of a specified power spectral density loading function. This is the type of analysis most often considered for random loading of structures. In the usual manner the frequency domain solution begins with an expansion of the modal equations of motions in terms of the homogeneous system eigenvalues and eigenvectors, normally referred to as the system undamped normal modes. For viscous like damping proportional to the system mass or stiffness the expanded equations of motion become a set of NR uncoupled single degree of freedom (DOF) oscillators.

\[
q_r(t) + 2 \beta \omega \omega_0^2 q_r(t) + \omega_0^2 q_r(t) = l_r(t),
\]

where \( q_r(t) \) is the modal DOF, \( \beta \) the equivalent viscous damping coefficient, \( \omega_0 \) the undamped normal mode frequency, and \( l_r(t) \) the generalized modal loading for the \( r \)th modal DOF. The structural modal motions, \( u \), are related to the modal DOF via the system eigenvectors as

\[
(\mathbf{q}) = \mathbf{e}_r \mathbf{u}.
\]

where \( \mathbf{e}_r \) is the \( r \)th mode eigenvector.
The generalized modal loading is determined from the applied nodal loading, \( f(S,t) \), via,
\[
l_i(t) = \int_0^T \phi_i(s) f(s,t) ds
\]
where the integration is over the structural surface, \( S \). For case of periodic loading, where the excitation occurs at distinct frequencies \( \omega \), the generalized loading takes the form
\[
L_i(\omega) = \int_0^T \phi_i(s) F(s) e^{j\omega t} ds
\]
where \( F(S) \) is the spatial variation in the applied periodic loading and \( \theta(S) \) is the relative phase variation across the structure. The explicit periodic variation of the form \( e^{i\omega t} \) is implied in the above equation via the use of upper case variables. The modal response, \( \Psi_i(\omega) \) to the periodic loading forms the cross-joint acceptance function for the structure, \( J(\omega)\).
For deterministic random loading in the form of a stationary Gaussian power spectral density (PSD) the one sided modal cross-spectral density between the \( m \)th and \( n \)th modal responses, \( G(\omega)\), is computed from
\[
G(\omega) = H(\omega) J(\omega)_m J(\omega)_n = \int_0^T \phi(s) PS\text{D}(s,s',\omega) \phi(s) ds ds'
\]
where \( \phi \) denotes the complex conjugate operator. The double surface integral is generally referred to as the cross-joint acceptance function for the structure, \( J(\omega)\). The cross-joint acceptance function describes how the input couples to the structure over its length or span. The corresponding displacement cross-spectral density function, \( G(\omega)\), takes the form
\[
G(\omega) = H(\omega) J(\omega)_m J(\omega)_n
\]
where the summation is over all the retained modal responses.

Stress modal functions are defined from the element stress displacement relationships and similar mathematical expressions result to describe the stress auto-power spectral density for an element. Often, the cross-modal terms are neglected; if they are computed, only the real part of the spectra has physical significance.

**Level I - Global Probabilistic**

A global probabilistic variation in the structure or loading is considered in Level 1 analysis. The global variation can occur in the following forms.

A. Stiffness Variation
\[
E = [K_0](1 + \lambda(x)) = [K_0]F
\]

B. Mass Variation
\[
\rho = [M_0](1 + \beta(x)) = [M_0]Z.
\]

C. Damping Variation

Viscous:
\[
\zeta = [C_0](1 + \zeta(x)) = [C_0]Z.
\]

Structural:
\[
\rho = [\rho_0](1 + \rho(x)) = [\rho_0]F
\]

These global variations or uncertainties scale the dynamic response variable mean values. The system eigenvectors, or mode shapes, scale as
\[
\Psi_i(\omega) = Z^{\frac{1}{2}} \Psi_i(\omega)
\]

The system eigenvalues, or frequencies squared, scale as
\[
\lambda_i = Z^{\frac{1}{2}} \lambda_i
\]
and the viscous proportional damping ratios scale as
\[
\beta_i = Z^{\frac{1}{2}} \beta_i
\]

where \( \Psi_i, \lambda_i, \) and \( \beta_i \) are, respectively, the \( i \)th mean eigenvector, eigenvalue, and viscous damping ratio.

D. Periodic Load Variation
\[
L_i(\omega) = \int_0^T \phi_i(s) F(s) e^{j\omega t} ds
\]

where \( F, \dot{F}, \) and \( \ddot{F} \) are respectively, the global stochastic variation in the periodic load magnitude, phase, and driving frequency and \( f(S, \theta(S)), \omega \) are respectively, the mean applied load, phase, and driving frequency.

E. PSD Variation
\[
PSD(\omega) = \{1 + R(\omega, t)\}^2 PSD(\omega) = \{1 + R(\omega, t)\} PSD(\omega)
\]

where \( R \) is the global stochastic variation in the random loading, and \( PSD(\omega) \) is the measured or most probable mean loading spectra. Here we do not distinguish between variation in PSD level, phase, or frequency on a global basis since such detail would not be unique.

The \( r \)th generalized modal response to random periodic loading of a structure with global stochastic variation of its mass, stiffness, and damping properties takes the form
\[
\Psi_r(\omega) = R(\omega, \beta_r) L_r(\omega)
\]

where \( L_r(\omega) \) is the Fourier transform of the \( r \)th generalized force of the mean loading and \( H_r(\omega, \beta) \) is the random oscillator transfer function for proportional viscous damping. An identical expression results for structural damping by replacing the transfer function with \( H(\omega, \beta) \).

The general form of the transfer function for viscous damping is
\[
H(\omega, \beta) = 2/(\rho' \omega^2 - C' \omega^2 + i2\beta \omega \omega^2)
\]

and for structural damping it is
\[
H(\omega, \beta) = 2/(\rho' \omega^2 - C' \omega^2 + i2\beta \omega \omega^2)
\]

The generalized load for the \( r \)th modal degree of freedom would take the first order form
\[
l_r(\omega) = 2^{1/2} \int_0^T \phi_r(s) f(s,t) \exp[-i\phi_r(s)] ds
\]

\[
-2^{1/2} \int_0^T \phi_r(s) f(s,t) \exp[-i\phi_r(s)] ds
\]

where \( F(\xi) \) is the zero mean random variation of the phase \( \phi \). Nodal responses such as displacements, velocities, and accelerations would then be expressed, respectively, as
\[ C(w) = \Re \left[ -\omega z \frac{\partial^2}{\partial t^2} \delta_{w_0} \langle Q_0(w) \rangle \right] \quad (22a) \]

\[ u_0(w) = \Re \left[ -\omega z \frac{\partial^2}{\partial t^2} \delta_{w_0} \langle Q_0(w) \rangle \right] \quad (22b) \]

and

\[ u_0(w) = \Re \left[ -\omega z \frac{\partial^2}{\partial t^2} \delta_{w_0} \langle Q_0(w) \rangle \right] \quad (22c) \]

where the \( \Re \) operator denotes the real part of the resultant. The element stresses would then be computed from the stress displacement relationships for the given element.

The major concern when introducing random excitation in the form of a power spectral density function (PSD) is how to introduce uncertainty in the structure and loading. In NESSUS the approach has been to assume that the uncertainty in the structure is independent of the uncertainty in the loading and vice versa. With this assumption, the modal cross-spectral density function becomes

\[ C_{\omega_{m\omega'}} = \Re \left[ 2^{1/2} \phi_{\omega_{m\omega'}} \langle \delta_{w_0} \rangle \langle Q_0(w) \rangle \right] \quad (23) \]

where the cross-joint acceptance function, \( J_{\omega_{m\omega'}}(w) \), does not exhibit global random variation due to the assumed independence in the structure and loading and as such takes the form as specified in Eq. (7).

The displacement cross-spectral density between the \( i \)th and \( j \)th nodes becomes

\[ C_{\omega_{i\omega j}} = \Re \left[ 2^{1/2} \phi_{\omega_{i\omega j}} \langle \delta_{w_0} \rangle \langle \delta_{w_0} \rangle \right] \quad (24a) \]

and likewise the velocities and accelerations are, respectively

\[ C_{\omega_{i\omega j}} = -\omega^2 \omega_{i\omega j} \quad (24b) \]

and

\[ C_{\omega_{i\omega j}} = \omega^4 I_{\omega_{i\omega j}} \quad (24c) \]

Stress modal functions are again defined from the element stress-displacement relationships; similar mathematical expressions result to describe the stress auto-power spectral density for an element. As with the deterministic case, the cross-modal terms are often neglected; if they are computed, only the real part of the spectra has physical significance.

**Level 2 - Full Uncertainty**

When considering full uncertainty throughout the structure, or parts thereof, and uncertainty in the loading with partially correlated random variables, explicit solutions to the dynamic response of the structure are no longer possible except in a few special cases. For the general case NESSUS synthesizes the partially correlated variables into uncorrelated modes (Wu et al., 1988). The system eigenvalues/eigenvectors are then computed, along with structural responses for a given loading, for each independent uncorrelated mode. The results are stored in a database and efficient probabilistic methods (Wu, 1987) are then used to construct dynamic response distribution functions for the response variables of interest.

**Time Domain Modeling**

For transient loading analyses the deterministic and full uncertainty cases are treated similarly. A Newmark \& Beta direct time integration scheme (e.g., Bathe, 1982) is employed to obtain the time history response of the structure. For the case of full uncertainty the partially correlated variables are synthesized into uncorrelated modes and each independent structural variation and loading case is solved independently. The analysis results are stored in a database and probabilistic methods are then used to construct response distribution functions.

**Dynamic Loading Functions**

The dynamic loading functions available in NESSUS are in the form of point harmonic loads, nodal harmonic pressures, surface distributed random loading, base excited point harmonic accelerations or acceleration power spectral density excitation, and explicit time history nodal loadings. Generic load models with multiple levels of progressive sophistication to simulate the composite load spectra that are induced in space propulsion system components representative of Space Shuttle Main Engines are being generated by Rockwell International Corporation, Rocketdyne Division engineers. Every effort has been made in the NESSUS code to accommodate a wide variety of possible loading scenarios. A major task underway is the integration of specified loading descriptions into the NESSUS load database.

**Uncertainty Characterizations of Dynamic Loads**

For static loading, it is straightforward to treat the amplitudes of the loads as random variables. The treatment of correlated random variables has been discussed in Walker (1981). Several NESSUS validation problems have been presented in the previously cited reference by Cruse et al.

When the loading is dynamic, the uncertainty of the loading becomes more difficult to model. In general, a dynamic loading can be treated as a stochastic process \( X(t) \). A stochastic process is a history containing an uncountable infinity of random variables, one for each \( t \). The statistical properties of \( X(t) \) are completely determined in terms of its nth-order distribution

\[ F(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\} \quad (25) \]

From the practical point of view, only certain averages are used. As an example, the statistics of a normal process are completely determined in terms of its mean and standard deviation.

For periodic loading, NESSUS treats the amplitude and phase as random variables. For stationary random loading, NESSUS treats the mean and the PSD as random variables. The PSD function is defined at several frequency points. At each point, the PSD can be described as a random variable. A special case is one in which the random variables are fully correlated so that only one random variable, i.e., a random scale factor of the PSD function, is needed. For multiple points random excitation, the correlations between each PSD function must be known.

For transient loading, it is possible to define a random variable for each time step if the loading can be described by using a reasonably small number of time steps. This model is more applicable when the loading function is relatively smooth and predictable over a period of time (i.e., the opposite of a white noise process in which the correlation is zero between any two instants). In general,
this model becomes impractical if the required number of time steps (thus the required number of random variables) is large. Therefore, simple stochastic process models (e.g., a Poisson process) which require only a few statistics would be more practical. If such models can be used to approximate the actual processes, then the uncertainty can be more easily characterized by treating the model statistics as random variables.

Validation Example

This example provides a sample of the NESSUS dynamics validation problems. In this example, a cantilever beam is subjected to random base excitation. The random variables include the modulus of elasticity (E), material density (ρ), length (L), thickness (t), damping factor (ξ), and constant acceleration PSD level (\(W_a\)). The PSD is modeled as a truncated white noise with cutoff frequency properly selected such that the random loading would excite, approximately, only the first mode. The statistics of the random variables are listed in the following table in which COV = coefficient of variation.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>COV</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>10 ( \times 10^6 ) lb/in(^2)</td>
<td>0.01</td>
<td>Lognormal</td>
</tr>
<tr>
<td>L</td>
<td>20.0 in</td>
<td>0.01</td>
<td>Normal</td>
</tr>
<tr>
<td>t</td>
<td>0.98 in</td>
<td>0.01</td>
<td>Normal</td>
</tr>
<tr>
<td>ξ</td>
<td>0.05</td>
<td>0.10</td>
<td>Lognormal</td>
</tr>
<tr>
<td>(W_a)</td>
<td>1.0 ( \text{in}^2/\text{sec}^3)rad</td>
<td>0.10</td>
<td>Lognormal</td>
</tr>
<tr>
<td>ρ</td>
<td>2.5 ( \times 10^{-4} ) lb/sec(^2)/in(^4)</td>
<td>0.02</td>
<td>Normal</td>
</tr>
</tbody>
</table>

Using a single degree of freedom model, the tip root-mean-square (RMS) displacement can be approximated as

\[
\text{RMS} = \sqrt{\frac{1.707L^{1.5}W_{a}^{1.5}}{E^{1.5}\xi}}
\]

(26)

The NESSUS probabilistic solution, in terms of the CDF, is shown in the figure below. In this figure, the MVFO (mean-value-first-order) solution employ only mean-based sensitivity data; the AMVFO (advanced MVFO) uses the information form MVFO and the re-computation of the response at a point based on probability considerations; the first iteration solution applies improved sensitivity data at the AMVFO points. The figure shows excellent agreement between the NESSUS probabilistic result and the 'exact' solution based on the Monte Carlo simulation (size 100,000) of the above equation. Note that there was a 3.9% difference between the NESSUS model and the analytical result at the 50% probability level. This difference has been used to adjust the 'exact' solution.

STOCHASTIC THERMOVISCOPLASTICITY

Introduction

The constitutive theory of thermoviscoplasticity is extremely general in nature (e.g., Walker, 1981), and is capable of describing a very wide variety of material behaviors, ranging from linearly viscous fluids to rate independent elastoplastic solids. Most thermoviscoplastic (TVP) theories or material models, though, are phenomenological, with a feature of such models being their dependence on phenomenological parameters that serve to describe gross material characteristics such as hardening, softening, and hysteresis. These parameters usually are not directly related to the actual physical properties of the material that give rise to its gross behavioral characteristics. Thus, these models correlate a material’s TVP behavior, but provide little or no insight into the active mechanisms that may be controlling the TVP behavior.

The purpose of the stochastic TVP material modeling effort within the PSAM project is to provide a sound basis for structural reliability modeling when stochastic TVP material behavior is to be included. The effort has been along two fronts. First, owing to the history (or path) dependent nature of TVP materials, and of phenomenological TVP material models that are typically used in calculations of structural response, a reasonable algorithm that preserves this path dependence has been proposed for calculating the evolution of the statistics of the TVP structure’s response. An important aspect of this problem is that the structure’s response statistics are directly related to the statistics of its material’s initial state. Consequently, the second part consists of describing the initial material state in terms of physical, or "primitive", material properties that are readily observable (such as grain sizes and shapes, dislocation density, etc.) and whose statistical characteristics are more readily estimated. Calculations of probabilistic TVP material response using such "primitive", or mechanistic, constitutive models will yield insight into how the statistics of a phenomenological model’s parameters are affected by the statistics of the readily observed "primitive" material variables.
Random Fields

The stochastic parameters of the probabilistic structural analysis problem may be split into two categories: the first being uncertainties in initial conditions; the second, uncertainties in external loading histories. The first category may contain uncertainties in initial material properties and uncertainties in the structure's initial geometry. In the most general case for random material behavior, the uncertainties in the initial material properties are described by random fields of these properties which initially span the structure. A random field is a function of three-dimensional space that possesses uncertainty of the parameters of the function, over the three-dimensional space (Liu et al., 1986).

The theory of random fields is fairly complex, but in practicality a finite element mesh is used to model the random field. Hence, the random fields can be reduced, as in the reference above, to sets of discrete, but correlated, random variables. To define the stochastic modeling problem, the uncertainties in the material properties need to be specified statistically. This is done by prescribing, as an initial condition, the joint probability density function (PDF) of these sets of (space-discrete) random material variables.

The second category of probabilistic variables may be introduced by prescribing uncertainties in the external loading history. In this case, the loading is a random function of time. Unlike the material modeling evolution, the stochastic nature of the loading can change at any time (e.g. random process). Similarly to before, the time axis can be discretized so that the loading history is prescribed through a set of discrete, correlated random variables with a specified joint PDF.

Probabilistic Evolution of TVP Behavior

It is taken as a basic tenet that, when a real TVP material is subject to a known loading history, its deformation mechanisms obey the laws of classical physics in a deterministic way, i.e., knowing its current history dependent microstructural state, and its current loading, the material “knows what to do,” and it does not deform by means of a random process. If the constitutive model was to be a random process, then this behavior would be akin to having the material characteristics, at a material point, change without the influence of any thermodynamic force. This is equivalent to saying that the current behavior is not related to a physical set of initial conditions, and therefore, this violates an engineering sensibility for material behavior.

In short, for a real TVP material (of known geometry subject to a known loading history), the randomness of its initial state, or its microstructural primitive variables, is the sole source of its randomness at later times. Stochastic TVP material models should also exhibit this characteristic. So, a stochastic TVP material is one whose initial material state is random, but which for a known initial geometry and a known loading history, responds deterministically in an evolutionary or path dependent sense (given a realized set of initial material parameters).

The geometry of the structure is another probabilistic initial value problem. If one were to allow the structural geometry to change in an incrementally stochastic way during some known loading history, then once again, there seems to be no guarantee that a realizable initial structure exists that would evolve in a deterministic way, under any realizable loading history, to reach such a "non-deterministically changed" geometric state.

These concepts lead one to define a stochastic TVP structure as one whose initial material state and whose initial geometric configuration are random, but which under a known loading history, given a realized set of initial material and geometric parameters, responds deterministically in an evolutionary or path dependent sense.

Proposed Solution Algorithm

Overview. First, define \( r = r(t) \) to be the response function of interest, where \( t \) is time. The function \( r \) is some subset of a complete realization, or analysis result. A realization is referred to herein as a physical occurrence, out of many possible occurrences. For example: \( r \) could represent the realized deflection history at some critical location in the TVP structure; it could represent the realized history of a stress component at some critical location; or, in the extreme case, it could represent the entire solution to the structural analysis.

For the moment though, consider the case where \( r(t) \) is a scalar quantity. The probabilistic problem is then to find, given the statistical descriptions of the uncertain (or random) input parameters, the statistical description of \( r = r(t) \). We choose, as this description, the response's cumulative probability distribution function (CDF), which will be denoted as \( c = c(r; t) \). Note that the response's CDF also evolves with time. If \( r \) is not a scalar, for example if it is an \( n \)-vector \( r(r_1, \ldots, r_n) \), then \( c \) is also an \( n \)-vector \( c = (c_1, \ldots, c_n) \), with \( c_i \) being the PDF of \( r_i \) \((i=1, \ldots, n)\).

Let \( k = (k_1, \ldots, k_N) \) represent the \( N \) discrete random parameters that describe the probabilistic TVP structure's random initial material and/or geometric state, and the random loading history to which it is subject, where the joint PDF of \( k \) is known. Also, denote the dependence of \( r \) on \( k \) as \( r(k; t) \). The solution procedure presented below is based on the previously cited, fast probability integration (FPI) method of Wu. Given \( r = r(k; t_0) \) and the joint PDF of \( k \) \((t_0 \text{ is some fixed time})\), the FPI algorithm provides an efficient and accurate method for the calculation of \( c = c(r; t_0) \). But, owing to the history dependence of the TVP structure, the function \( r = r(k; t_0) \) is difficult (or at least potentially very time consuming) to calculate.

During the calculation of \( c = c(r; t_0) \), the FPI code will need to evaluate \( r = r(k; t_0) \) for many different values of \( k \). Conceptually then, for each evaluation of \( r \), a deterministic finite element analysis, for example, based on the particular needed realization of \( k \), needs to be performed in an incremental fashion integrating from time \( 0 \) to time \( t_0 \). Obviously, this is unacceptable if too many evaluations of \( r \) are needed.

A reasonable way to avoid the calculation of so many realizations is to construct an approximate analytical form for \( r = r(k; t_0) \). The approach taken here is quite simple and, in fact, is the same that has been described by Wu and Wirsching (1984). The idea is to pick several
(say m) realizations of k (call them k_i, i=1,...,m), and calculate out to the latest time of interest the m deterministic TVP boundary value problems based on each of the k_i's in turn. Then, at t_0, a polynomial is fit to the m (k_i,r_i≡r(k_i;t_0)) points, which provides an approximate analytical form for r(r(k);t_0) that the FPI program can quickly evaluate when calculating c=c(r;t_0). Polynomial fits made at various other (fixed) times can then be used by the FPI program to calculate the evolution of c=c(r;t) in time.

**Example of Evolutionary Distributions**

This example seeks to illustrate the nature of stochastic simulation of evolutionary relationships, such as occurs for TVP material response. The example will use pre-selected analytical response-time curves, which is done so that the exact expression for c=c(r;t) can be obtained and compared to the results of the FPI-based algorithm. The basic idea of the algorithm is to fit approximate response functions to evolved response-time curves at different time points, and to use the FPI algorithm to define the CDF results at user-selected time points. It will be seen that the algorithm is able to reproduce the exact results with excellent agreement. It will also be seen that the statistical nature of the answer is also evolving with time. In the next sub-section, this FPI-based algorithm will be applied to a simulated TVP material.

Let r be taken as a scalar (i.e., r=(r_1) and c=(c_1)) and let N=1 (i.e., there is only one random variable, k=(k_1)). For simplicity, denote (r_1) as r, (c_1) as c, and (k_1) as k. The PDF taken for k is

\[ p(k) = 6k(k-1), \quad 0 < k < 1. \]  

(27)

Now, choose, for example, five different k values (m=5) to span the range of interest, i.e., k=0.005, 0.250, 0.500, 0.750 and 0.995, and for each of these, say that a TVP boundary value problem was calculated out to a time of t=4 via incremental finite element analysis, which yielded the five traces of r vs t shown in the figure below.

**FIG. 6(a) REALIZATIONS OF RESPONSE VS. TIME**

The traces shown in the figure are actually plots of the response-time curve

\[ r = t. \]  

(28)

but, in general, the exact analytical form for these curves would not be known and is used here only for the sake of demonstration. Next, at some time of interest t=t_0, the five (k,r) points are used to define a fourth-order polynomial, i.e.,

\[ r = a_0 + a_1k + a_2k^2 + a_3k^3 + a_4k^4, \]  

(29)

as the needed approximate analytical form for r=r(k;t_0).

For example, at t_0=2 one obtains,

\[ a_0 = 0.6929735040, \quad a_1 = 0.2416176906, \quad a_2 = 0.0136692352. \]  

(30)

and at t_0=4,

\[ a_0 = 1.378772670, \quad a_1 = 1.020432321. \]  

(31)

The three equations above were used by the FPI program to obtain the CDF's c(r;2) and c(r;4) as indicated, respectively, by the '+'s and squares in the figure below; the solid and dashed curves in the figure are the exact analytical solutions described above. As can be seen, this approach works quite well.

Although no figures are given here as evidence, an important aspect of the solution is that the expected value of r, when plotted against t, does not correspond deterministically to produce the expected response as a function of time. The traces shown in the figure are actually plots of the response-time curve

\[ r = t. \]  

(28)

However, the exact analytical form for these curves would not be known and is used here only for the sake of demonstration. Next, at some time of interest t=t_0, the five (k,r) points are used to define a fourth-order polynomial, i.e.,

\[ r = a_0 + a_1k + a_2k^2 + a_3k^3 + a_4k^4, \]  

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Although no figures are given here as evidence, an important aspect of the solution is that the expected value of r, when plotted against t, does not correspond deterministically to produce the expected response as a function of time.
to obtain this additionally required information by using, at \( t_0 \), one of the realizations as a base from which to perturb, then this would be equivalent to having the material deform by a random process. As was previously discussed, this is an unacceptable situation. Additional research is needed if a solution procedure more efficient than the one presented above is to be found.

Finally, it should be noted that perturbation techniques can be used for history insensitive nonlinear problems (such as the finite elasticity of a perfect rubber band), since it is irrelevant by what path one arrives at the current state. In such a case, a sensible algorithm would be to track a single realization \( r=r(k;t) \), e.g., one corresponding to the mean value of \( k \). Then, at some \( t_0 \) where \( c=c(r;t_0) \) is desired, the \( r \)-solution obtained from the mean value of \( k \) could be used as a base from which to iterate (e.g., using the equations of finite elasticity) toward and to any new \( r=r(k;t_0) \) that the FPI program may require.

**Probabilistic TVP Tension Test**

As an example of probabilistic TVP material modeling, the above procedure is applied to a high-temperature (or creep) tension test. The creep model used is based on the mechanism of grain boundary sliding with grain boundary diffusional accommodation, and is similar in spirit to the model of Raj and Ashby (1971).

The polycrystalline material is idealized as being a two-dimensional array of hexagonal grains, as depicted in Part (a) of the figure below, with the grain's current size and shape being defined by the lengths and \( d \) and \( j \). As indicated in the figure, the assemblage is subjected to an overall tensile true stress \( \sigma_2 \), with the other overall in-plane stress components being zero. Under such a loading, the deformation is assumed to occur by grain boundary sliding in the mode shown by Part (b) of the figure, where the left and right grains are being "pulled in" by the "void" on the grain boundary BE. In this state, the grain boundaries AB, BC, DE and EF have (local) compressive stresses acting normally across them, and the grain boundary BE has a tensile normal stress acting across it.

The local normal stresses lead to diffusional mass flux along the grain boundaries in the senses indicated by the arrows, and this removes mass from along the four oblique grain boundaries and deposits it in the "void" of grain boundary BE. In this way, the deformation is diffusional accommodated.

\[
\begin{align*}
\sigma_1 &= 0 \\
\sigma_2 &= 0 \\
d &= d\varepsilon, \quad \varepsilon = \varepsilon^e + \varepsilon^r
\end{align*}
\]

![FIG. 7(a)](image-url) **FIG. 7(a) IDEALIZED TWO-DIMENSIONAL MICROSTRUCTURE**

In a method similar to that of Raj and Ashby (1971), the grain boundary mass flux is taken proportional to \( \frac{\partial a_1}{\partial S} \) (where \( a_1 \) is the normal stress acting across the grain boundary, and \( S \) is the distance along the grain boundary). This, coupled with mass conservation allows one to find, for a given overall inelastic strain rate, \( \varepsilon^e \), the corresponding distribution of \( a_1 \) along the grain boundaries. Force equilibrium applied to certain grain cross-sections, along with the assumption that the grain boundaries support no shear stress, then provides the corresponding value of \( a_1 \).

The procedure (Harren, 1988) yields, for this orthotropic deformation mode, the simple flow rule

\[
\dot{\varepsilon}^i = \frac{24\lambda\beta}{d^3f(A)} a_1, \quad \dot{\varepsilon}^e = -\varepsilon, \quad \varepsilon_1 = 0.
\]

where \( \varepsilon_i \) (i=1,2,3) are the three inelastic logarithmic strain rates along each of the coordinate axes (there are no shear strains), \( \dot{\varepsilon}^i \) is the current strain rate, \( \sigma_1 \) is the normal stress acting across the grain boundary, \( A \) is the current grain aspect ratio (\( A = 1 \) corresponds to the grains being regular hexagons). In the previous two equations, the current size and shape of the grains are determined from the simple evolution equations

\[
\dot{1} = \dot{d}, \quad \dot{d} = d\varepsilon, \quad \varepsilon = \varepsilon^e - \varepsilon^r
\]

where \( \varepsilon_b \) and \( \varepsilon^r \) (i=1,2,3) are, respectively, the total and elastic logarithmic strain rates. The diffusion coefficient \( D \) is given by

\[
D = \frac{D_s b}{RT} \quad \text{and} \quad D_s = D_s' \exp\left(-\frac{Q_b}{RT}\right) \quad R = N_a, B
\]

where \( b \) is Boltzmann's constant, \( T \) is absolute temperature, and \( N_a \) is Avogadro's number. The grain boundary thickness \( b \) is assumed to remain constant with time, as are the constants \( \Omega \) (the molecular volume), \( D_s' \) (the limiting grain boundary diffusivity) and \( Q_b \) (the grain boundary diffusion activation energy).
In the current probabilistic problem, both the deformation and the material properties of the tensile bar are taken as being uniform with respect to space. The T-1255K case is considered, and all of the constants are as above, except that now $E$ and $Q_b$ are random, i.e.,

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} f_r, \quad Q_b = q_r.$$  \hspace{1cm} (41)

Also, the effective stress is taken as the response, i.e., $\sigma = r$. The variables $E$ and $Q_b$ are assumed to be statistically independent, and each has a truncated normal distribution as its PDF. The mean value of $E$, i.e., $\mu(E)$, is chosen as 150 GPa, and $\mu(Q_b)$ is chosen as 72.5 kJ/mole. The standard deviations for the problem are $\sigma(E) = 10$ GPa and $\sigma(Q_b) = 0.5$ kJ/mole. The cut-off values for both of these PDF's are taken to occur at plus and minus three standard deviations from the mean.

In order to construct accurate representations of $r = r(k, q_0)$, 25 separate realizations were calculated, each corresponding to one of the possible combinations of $E = 120.5, 135.0, 150.0, 165.0$ and 179.5 GPa, and $Q_b = 71.05, 71.75, 72.50, 73.25$ and 73.95 kJ/mole. The 25 resulting stress-strain curves are shown in the next figure, and at each of the three different time levels shown, i.e., $t_0 = 10.0$ sec ($\epsilon_t = 8.3 \times 10^{-4}$), 67.5 sec ($\epsilon_t = 5.6025 \times 10^{-3}$) and 300. sec ($\epsilon_t = 2.49 \times 10^{-2}$), the 25 $(k, r)$ points fit to a fourth order, two-dimensional Lagrangian polynomial, i.e., a polynomial whose individual terms are

$$c = a_0 + a_1 \epsilon + a_2 \epsilon^2 + a_3 \epsilon^3 + a_4 \epsilon^4,$$  \hspace{1cm} (42)

where $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, and $k = 1, 2, 3, 4, 5$. The three resulting polynomials were then used by the FPI program to obtain the CDF's $c(\epsilon_t; 10.0$ sec), $c(\epsilon_t; 67.5$ sec), and $c(\epsilon_t; 300.0$ sec), which are shown in the following figures. The probability levels of $c$ are expressed in terms of the standard normal unit $u$, i.e.,

$$c = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{\mu}{\sigma} \right) \right] \quad \text{with} \quad u = \frac{\sigma(\epsilon_t; q_0) - \mu(\epsilon_t; q_0)}{\sigma(\epsilon_t; q_0)},$$  \hspace{1cm} (43)

where "erf" is the error function (a normal distribution plots as a straight line on such a graph). As is seen, the character of the distribution changes in time: at $t_0$.
-10.0 and 300.0 sec, fairly high curvatures are exhibited in the distributions' tails, while at $t_0 = 67.5$ sec the curvature is less pronounced. This is due to the fact that, at $t_0 = 10.0$ sec, $E$ is dominant (i.e., $\phi_0 = 0$) and at $t_0 = 300.0$ sec $Q$ is dominant ($\phi_0 = 0$), while at $t_0 = 67.5$ sec neither dominates.

It has been determined that the stochastic aspects of history dependent problems precludes a direct application of the perturbation method. This stems from the fact that the current solution depends on the integrated history of all previous states; e.g., a current perturbation is not likely to have a relationship to a physical initial state. Additionally, the randomness of a material at some solution state depends on its initial randomness, not its current state. The proposed algorithm for stochastic evolution of material response curves has been demonstrated for relatively few response histories.

The current NESSUS implementation for nonlinear analysis will make use of multiple, evolving solutions, each of which is related to random material properties, in order to compute the distribution of response variables for these problems. Further work is required to establish efficient means for nonlinear analysis of stochastic structures with spatially varying stochastic-TVP histories.

REFERENCES


