A Study on Stability and Response Analysis of a Nonlinear Rotor System with Mass Unbalance and Side Load

TING NUNG SHIAU
Institute of Aeronautics and Astronautics
National Cheng Kung University
Tainan, Taiwan, R.O.C.

JON LI HWANG and YUAN BIN CHANG
Group of Structural Mechanics
ARL/AIDC/CSIST, Taichung, Taiwan, R.O.C.

ABSTRACT

The stability of steady state synchronous and nonsynchronous response of a nonlinear rotor system supported by squeeze-film dampers is investigated. The nonlinear differential equations which govern the motion of rotor bearing system are obtained by using the Generalized Polynomial Expansion Method. The steady state response of system is obtained by using the hybrid numerical method which combines the merits of the harmonic balance and collocation methods. The stability of system response is examined using Floquet-Liapunov theory. Using the theory, the performance may be evaluated with the calculation of derivatives of nonlinear hydrodynamic forces of the squeeze-film damper with respect to displacement and velocity of the journal center. In some cases, these derivatives can be expressed in closed form and the prediction of the dynamic characteristic of the nonlinear rotor system will be more effective. The stability results are compared to those using a direct numerical integration method and both are in good agreement.

NOMENCLATURE

- $E, E_w$: nonlinear hydrodynamic forces in the $Y$ and $Z$ directions
- $F_x$: side load
- $[K_s]$: system generalized stiffness matrices of shaft
- $[K_{sy}], [K_{sz}]$: system generalized stiffness matrices of bearing
- $k$: stiffness of centering spring
- $[L(T)]$: transition matrix
- $[M], [G]$: system generalized mass and gyroscopic matrix, respectively
- $m$: mass of rigid disk
- $N_p$: number of terms of polynomial
- $N_u$: total number of harmonic terms retained
- $R, C_r, L$: radius, radial clearance, and width of squeeze-film damper, respectively
- $r$: radial displacement of journal center
- $(v, w)$: lateral deflection in $(Y, Z)$ directions
- $(u_x, u_y)$: steady state response of nonlinear system
- $(B, \Gamma)$: rotating deflection in $X-Z$ and $X-Y$ planes
- $\delta$: nondimensional mass unbalance ($\delta = \epsilon / C_s$)
- $\epsilon, \epsilon'$: nondimensional radial displacement and radial velocity of journal center ($\epsilon = r / C_r$)
- $\Lambda$: eigenvalues of transition matrix
- $\mu$: lubricant viscosity
- $\rho$: mass density per unit volume of the shaft
- $\phi$: precession angle of journal center
- $\psi$: precession velocity of journal center
- $\omega, \Omega$: whirl speed and rotating speed

Presented at the International Gas Turbine and Aeroengine Congress and Exposition
Cologne, Germany June 14, 1992
This paper has been accepted for publication in the Transactions of the ASME Discussion of it will be accepted at ASME Headquarters until September 30, 1992

Copyright © 1992 by ASME
INTRODUCTION

Squeeze film dampers (SFD) have been widely used in modern rotating machines to dampen rotor motion and it has been verified that the rotor vibrational amplitudes can be reduced and the system stability can be improved with squeeze-film dampers (Mohan and Hahn, 1974; Gunter et al., 1977; Rabinowitz and Hahn, 1983; Chen, 1987). Because of the complicated hydrodynamic behavior of a squeeze-film damper, it has become a special part of rotor dynamics.

Several investigators (Mohan and Hahn, 1974; Gunter et al., 1977; Hahn, 1979; Taylor and Kumar, 1980; Greenhill and Nelson, 1982; Rabinowitz and Hahn, 1983) have shown that the steady state response will be a centric circular synchronous motion, if a rotor system is symmetric and vertical, or if a rotor system is preloaded such that the rest journal's center is coincident with the bearing's center. The stability of the centric circular synchronous motion can be studied by firstly perturbing the equations of motion about the steady state solution and secondly solving the eigenvalues problem of the perturbed system. It has been shown by researchers (Hahn, 1979; Athre et al., 1982; Greenhill and Nelson, 1982; Chen, 1987) that the steady state centric circular motion is unstable if one of the real part of eigenvalues is positive. Another method for studying the stability of steady state circular motion is to search for the convergent passage by direct numerical integration (Taylor and Kumar, 1980).

The steady state periodic response of a rotor system with squeeze-film dampers may not be a centric circular motion due to the effects of non-synchronous excitations and/or unsymmetrical configuration of a rotor system. Many researchers investigated such periodic response using various methods, such as direct numerical integration (Cookson and Kossa, 1979), perturbation method (Pan and Tonnessen, 1978), trigonometric collocation method (TCM) and harmonic balancing method (HBM) (Saito, 1985; Nataraj and Nelson, 1989; Shiau and Jean, 1990; Jean and Nelson, 1990). However, there are lack of the stability analysis of such periodic response.

The stability of nonlinear steady state response and the occurrence of non-synchronous response will be studied in this paper. For non-centric circular synchronous motion, the perturbed dynamic behavior will be governed by a set of linear ordinary differential equations with periodic coefficients. The Floquet-Liapunov theory is used to examine the stability of the periodic system. An improved numerical integration method (Friedmann et al., 1977) is used to calculate the Floquet transition matrix which is used to determine system stability.

For the analysis of stability, the derivatives of nonlinear squeeze-film forces with respect to journal displacement and velocity will be employed. With the application of short bearing theory, the derivatives of nonlinear squeeze-film forces can be expressed in closed form. A simple example of rigid rotor supported by squeeze-film damper is firstly employed to demonstrate the stability of the nonlinear periodic response as well as the occurrence of 1/2-subharmonic whirl motion. In addition, the nonlinear response of a flexible rotor system, which consists of multiple rigid discs and bearing supports, are calculated by using the hybrid numerical method (Hwang and Shiau, 1991) and the corresponding stability will be examined by using the present algorithm.

DERIVATION OF GOVERNING EQUATIONS

A rigid rotor carried in a squeeze-film damper with linear isotropic centering spring and a rotor mass, m, concentrated at the axial center of journal, is shown in Figure 1. The equations of motion of the rotor system can be expressed in a rotating frame (Mohan and Hahn, 1974; Cookson and Kossa, 1979; Taylor and Kumar, 1980) as follow :

\[ m(\ddot{r} - \omega^2 \phi) + kr = m\omega^2 \cos(\omega t - \phi) - F_s \cos(\frac{\pi}{2} - \phi) + F_s \]

\[ m(\dot{\phi} + 2\omega \dot{\phi}) = m\omega^2 \sin(\omega t - \phi) - F_s \sin(\frac{\pi}{2} - \phi) + F_s \]  

(1)

where the parameters are defined in the nomenclature and the rotating speed \( \omega \) is constant. Alternatively, the equations of motion can be expressed in the inertia frame as

\[ m\ddot{u} + ku = F_u + m\omega^2 \cos(\omega t) \]

\[ m\ddot{w} + kw = F_w + m\omega^2 \sin(\omega t) - F_s \]

(2)

It is noted that the forces generated by the squeeze-film damper in the rotating frame \( (F_s, F_u) \) and/or in inertia frame \( (F_s, F_u) \), are nonlinear functions of the displacement and velocity of the journal center.

For the flexible rotor system, the Generalized Polynomial Expansion Method (GPEM) proposed by Hwang and Shiau (1991) is employed. It describes that the deflections of the flexible shaft can be expressed as functions of axial coordinate \( x \), flexible shaft and time \( t \), as follow :

\[ v(x, t) = \sum_{n=1}^{N_v} a_n(t) x^{n-1} \]

\[ w(x, t) = \sum_{m=1}^{N_w} b_m(t) x^{m-1} \]

\[ B(x, t) = - \frac{\partial w(x, t)}{\partial x} = - \sum_{m=2}^{N_w} (m - 1) x^{m-2} b_m(t) \]

(3a)
\[ \Gamma(x,t) = \frac{\partial u(x,t)}{\partial x} = \sum_{n=2}^{N_p} (n-1) x^{n-2} a_n(t) \]

where the \( a_n(t) \) and \( b_n(t) \) are named generalized coordinates and the integer \( N_p \) is the total number of polynomials. Using the Lagrangian approach, the equations of motion can be expressed as

\[
\begin{bmatrix}
M & 0 \\
0 & M
\end{bmatrix}
\begin{bmatrix}
\ddot{\mathbf{a}} \\
\ddot{\mathbf{b}}
\end{bmatrix}
+ \begin{bmatrix}
C_{yy} & C_{yx} \\
C_{xy} & C_{xx}
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{a}} \\
\dot{\mathbf{b}}
\end{bmatrix}
+ \begin{bmatrix}
K_{yy} & K_{yx} \\
K_{xy} & K_{xx}
\end{bmatrix}
\begin{bmatrix}
\mathbf{a} \\
\mathbf{b}
\end{bmatrix}
+ \begin{bmatrix}
E^a_{1} \\
E^b_{1}
\end{bmatrix}
+ \begin{bmatrix}
E^a_{2} \\
E^b_{2}
\end{bmatrix}
\]

\[ \text{(3b)} \]

where

\[ \mathbf{a} = \{ a_1, a_2, \ldots, a_N \} \]

\[ \mathbf{b} = \{ b_1, b_2, \ldots, b_N \} \]

Applying the numerical technique by Hwang and Shiau (1991), the steady state response of the rotor system can be obtained and expressed by a set of finite terms of trigonometric functions. It should be noted that the linear and nonlinear forces due to the squeeze-film damper are symbolically included in the first and the second term on the right hand side of equation (3b), respectively.

**NONLINEAR SQUEEZE-FILM FORCES AND THEIR DERIVATIVES**

To obtain the forces of squeeze-film damper, the short bearing approximation is used because most of squeeze-film dampers are of low \( L/D \) ratio (Mohan and Hahn, 1974; Gunter et al., 1977), with \( L \) the width of bearing and \( D \) the diameter of journal. The nonlinear forces of squeeze-film damper can be obtained by integrating the effective pressure distribution determined from solving the Reynolds equation, along the bearing surface. And these nonlinear forces can be expressed as

\[
\begin{align*}
\{ F_r \} &= -\frac{\mu RL^3}{C_f^2} \int_{\theta_1}^{\theta_2} \left[ \cos \theta + c \sin \theta \right] \left[ \cos \theta + c \sin \theta \right]^3 d\theta \\
\{ F_\phi \} &= -\frac{\mu RL^3}{C_f^2} \int_{\theta_1}^{\theta_2} \left[ \cos \theta + c \sin \theta \right]^3 d\theta \\
\{ F_v \} &= -\frac{\mu RL^3}{C_f^2} \int_{\theta_1}^{\theta_2} \left[ \cos \theta + c \sin \theta \right] d\theta \\
\{ F_b \} &= -\frac{\mu RL^3}{C_f^2} \int_{\theta_1}^{\theta_2} \left[ \cos \theta + c \sin \theta \right]^3 d\theta
\end{align*}
\]

\[ \text{(4)} \]

or

\[
\begin{align*}
\{ F_r \} &= -\frac{\mu RL^3}{C_f^2} \left\{ \cos \theta_1 + c \sin \theta_1 \right\} \\
\{ F_\phi \} &= -\frac{\mu RL^3}{C_f^2} \left\{ \cos \theta_1 + c \sin \theta_1 \right\}^3 \\
\{ F_v \} &= -\frac{\mu RL^3}{C_f^2} \left\{ \cos \theta_1 + c \sin \theta_1 \right\} d\theta \\
\{ F_b \} &= -\frac{\mu RL^3}{C_f^2} \left\{ \cos \theta_1 + c \sin \theta_1 \right\}^3 d\theta
\end{align*}
\]

\[ \text{(5)} \]

where the coefficients \( A_{ij}^l \) are defined as follow

\[
A_{ij}^l = \int_{\theta_1}^{\theta_2} \left[ \sin \theta + c \cos \theta \right] \left[ \sin \theta + c \cos \theta \right]^3 d\theta,
\]

and \(( l ) \equiv \int_{\theta_1}^{\theta_2} \left[ \sin \theta + c \cos \theta \right] d\theta\). It is noted that the squeeze film forces will generally depend on the motion of journal center which is usually implicit function of time, as shown in equations (4) and (5). In addition, if the motion orbit of system is synchronous circular and the oil film is fully cavitated (i.e. \( \pi \)-film model), equation (6) can be approximately expressed as

\[
\begin{align*}
F_r &= -\frac{\mu RL^3}{C_f^2} \left[ \cos \theta + c \sin \theta \right] d\theta \\
F_\phi &= -\frac{\mu RL^3}{C_f^2} \left[ \cos \theta + c \sin \theta \right]^3 d\theta
\end{align*}
\]

\[ \text{(6)} \]

The dimensionless variables shown in equation (5) are given by

\[
\epsilon = r/C_r, \quad V = v/C_r, \quad W = w/C_r
\]

\[
\theta_f = \theta_f + \phi, \quad \theta_{f1} = \theta_f + \phi, \quad \theta_{f2} = \theta_f + \phi
\]

\[
\phi = \tan^{-1}\left( \frac{W}{V} \right)
\]

Moreover, the integration limit can be generally expressed as

\[
\theta_{f2} = \theta_{f1} + \text{constant}
\]

which is the case of general application. For example, the constant value is \( 2\pi \) and \( \theta_{f2} \) can be any constant value for \( 2\pi \)-film bearing model. However if the constant value is \( \pi \) and \( \theta_{f2} \) will be of the form

\[
\theta_{f2} = \tan^{-1}\left( \frac{W}{V} \right)
\]

\[ \text{(11)} \]

which is the \( \pi \)-film bearing model.

The derivatives of nonlinear squeeze-film forces in either rotating frame or inertia frame can be obtained by differentiating equations (4) and (5) with respect to journal displacement and velocity. They are of the form:

(a) rotating frame case

\[
\begin{align*}
\frac{\partial}{\partial \theta} \left\{ F_r \right\} &= 0 \\
\frac{\partial}{\partial \theta} \left\{ F_\phi \right\} &= 0 \\
\frac{\partial}{\partial \theta} \left\{ F_v \right\} &= 0 \\
\frac{\partial}{\partial \theta} \left\{ F_b \right\} &= 0
\end{align*}
\]

\[ \text{(12)} \]

(b) inertia frame case

\[
\begin{align*}
\frac{\partial}{\partial \theta} \left\{ F_r \right\} &= -2B \left\{ \phi A_{f1}^1 + 3\phi A_{f2}^1 + 3\phi A_{f3}^1 \right\} \\
\frac{\partial}{\partial \theta} \left\{ F_\phi \right\} &= -2B \left\{ \phi A_{f2}^2 + 3\phi A_{f2}^1 + 3\phi A_{f3}^2 \right\} \\
\frac{\partial}{\partial \theta} \left\{ F_v \right\} &= -2B \left\{ \phi A_{f3}^3 + 3\phi A_{f3}^2 + 3\phi A_{f3}^1 \right\} \\
\frac{\partial}{\partial \theta} \left\{ F_b \right\} &= -2B \left\{ \phi A_{f3}^4 + 3\phi A_{f3}^3 + 3\phi A_{f3}^2 \right\}
\end{align*}
\]

\[ \text{(13)} \]
\[
\frac{\partial}{\partial t} \begin{bmatrix} F_v \\ F_w \end{bmatrix} = -B \begin{bmatrix} A_{00}^{(0)} \\ A_{00}^{(1)} + 3 \phi \phi \epsilon_{A_{00}^{(1)} + 3 \phi \phi \epsilon_{A_{00}^{(1)}}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \end{bmatrix} \\
\frac{\partial}{\partial \phi \phi} \begin{bmatrix} F_v \\ F_w \end{bmatrix} = -B \begin{bmatrix} A_{00}^{(0)} \\ A_{00}^{(1)} + 3 \phi \phi \epsilon_{A_{00}^{(1)} + 3 \phi \phi \epsilon_{A_{00}^{(1)}}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \end{bmatrix}
\]

where

\[ B = \frac{\mu R L^3 \Omega}{C^2} \]

and

\[ L_{ij}^{(m,n)} = \begin{bmatrix} \sin \theta_i \sin \theta_j & \cos \theta_i \sin \theta_j & \sin \theta_i \cos \theta_j \end{bmatrix}^T \begin{bmatrix} \cos \theta_i \sin \theta_j & \sin \theta_i \sin \theta_j & \cos \theta_i \cos \theta_j \end{bmatrix} \]

(b) Inertia frame case

\[ \frac{\partial}{\partial V} \begin{bmatrix} F_v \\ F_w \end{bmatrix} = -B \begin{bmatrix} A_{00}^{(0)} \\ A_{00}^{(1)} + 3 \phi \phi \epsilon_{A_{00}^{(1)} + 3 \phi \phi \epsilon_{A_{00}^{(1)}}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \end{bmatrix} \]

\[ \frac{\partial}{\partial \phi} \begin{bmatrix} F_v \\ F_w \end{bmatrix} = -B \begin{bmatrix} A_{00}^{(0)} \\ A_{00}^{(1)} + 3 \phi \phi \epsilon_{A_{00}^{(1)} + 3 \phi \phi \epsilon_{A_{00}^{(1)}}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \\ A_{00}^{(2)} + 3 \phi \phi \epsilon_{A_{00}^{(2)}} \end{bmatrix} \]

The formulation of these derivatives shown in equations (12)—(26) will be employed in the stability analysis. The nonlinear forces and their derivatives are expressed in rotating frame because the expressions are simple. However, for studying the stability and the response of system, it is more convenient to use the expressions in fixed frame. Also, it should be noted that the use of \( \sigma \)-film model is only for convenience.

**STABILITY OF STEADY STATE RESPONSE**

The steady state solution of equation (2) can be approximately described (Saito, 1985; Nataraj and Nelson, 1990; Shinu and Jean, 1990; Jean and Nelson, 1990) by the form

\[ v = v_0 + \sum_{i=1}^{N_v} (w_i \cos \omega_i t + v_i \sin \omega_i t) \]

where \( \omega_i = \omega_0 + \sum_{i=1}^{N_w} (w_i \cos \omega_i t + v_i \sin \omega_i t) \)

where \( \omega_0 \) is an integer number. To examine the stability of the steady state solution, the motion is perturbed and the resulting motion is expressed as

\[ v = v_{ss} + \delta v \]
\[ w = w_{ss} + \delta w \]

And the perturbed squeeze-film forces can be expressed as

\[ \{ F_v \} \approx \{ F_{v,ss} \} + \frac{\partial F_v}{\partial \delta v} \delta v + \frac{\partial F_v}{\partial \delta w} \delta w \]
\[ \{ F_w \} \approx \{ F_{w,ss} \} + \frac{\partial F_w}{\partial \delta v} \delta v + \frac{\partial F_w}{\partial \delta w} \delta w \]

where the subscript "ss" denotes the steady state solution and the coefficient matrices can be obtained from equations (18)—(21). Substituting equations (27)—(29) into equation (2), the perturbed equations are of the form:

\[ \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \delta v \\ \delta w \end{bmatrix} + \begin{bmatrix} \frac{\partial F_v}{\partial \delta v} & \frac{\partial F_v}{\partial \delta w} \\ \frac{\partial F_w}{\partial \delta v} & \frac{\partial F_w}{\partial \delta w} \end{bmatrix} \begin{bmatrix} \delta v \\ \delta w \end{bmatrix} = \{ 0 \} \]

\[ \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \delta v \\ \delta w \end{bmatrix} + \begin{bmatrix} -k \frac{\partial^2 F_v}{\partial \delta v^2} & -k \frac{\partial^2 F_v}{\partial \delta w^2} \\ -k \frac{\partial^2 F_w}{\partial \delta v^2} & -k \frac{\partial^2 F_w}{\partial \delta w^2} \end{bmatrix} \begin{bmatrix} \delta v \\ \delta w \end{bmatrix} = \{ 0 \} \]
The stability of nonlinear steady state response is governed by equation (30) which is a set of homogeneous ordinary differential equations. For the special case of centric circular synchronous motion, i.e.,
\[ \omega_1 = \Omega, \quad v_0 = w_0 = 0 \]
\[ v_{ci} = v_{si} = w_{ci} = w_{si} = 0 \quad \text{for} \ i \neq 1 \]
\[ v_{ci} = w_{si}, \quad v_{si} = -w_{ci} \]

or equivalent to
\[ r = \text{constant}, \quad \dot{r} = 0 \]
\[ \dot{\phi} = \Omega = \text{constant} \]

The solution of equation (30) can be written in rotating frame. That yields a set of homogeneous ordinary differential equations with constant coefficients. Thus, the stability problem can be examined by solving the eigenvalues of a constant matrix (Hahn, 1979; Athre et al., 1982; Greenhill and Nelson, 1982; Chen, 1987).

In general, the coefficient matrices shown in equation (30) are periodic functions of time with period 2π/ω1. The stability problem can be examined by solving the eigenvalues of the transition matrix of the periodic system. Applying the Floquet-Liapunov theory, the steady state periodic response is stable only when the maximum absolute value of the eigenvalues of transition matrix is smaller than one, i.e.,
\[ |\lambda|_{\text{max}} < 1.0 : \text{Stable} \]
\[ |\lambda|_{\text{max}} \geq 1.0 : \text{Unstable} \]  \hspace{1cm} (33)

For the convenience of analysis, equation (30) is cast into the first order form
\[ \{\delta \dot{q}\} = [S]\{\delta q\} \]
\[ \text{where} \]
\[ \{\delta \dot{q}\} = \{\delta \dot{v}, \delta \dot{w}, \delta \dot{\phi}, \delta \dot{\omega}\}^T \]
\[ [S] = \frac{1}{m} \begin{bmatrix}
\frac{\partial P_r}{\partial v} & \frac{\partial P_r}{\partial w} & \frac{\partial P_r}{\partial \phi} & \frac{\partial P_r}{\partial \omega} - k & \frac{\partial P_m}{\partial v} \\
\frac{\partial P_r}{\partial v} & \frac{\partial P_r}{\partial w} & \frac{\partial P_r}{\partial \phi} & \frac{\partial P_m}{\partial \omega} - k & \frac{\partial P_m}{\partial v} \\
\frac{\partial P_r}{\partial v} & \frac{\partial P_r}{\partial w} & \frac{\partial P_r}{\partial \phi} & \frac{\partial P_m}{\partial \omega} - k & \frac{\partial P_m}{\partial v} \\
0 & 0 & 0 & 0 & 0 \\
0 & m & 0 & 0 & 0 
\end{bmatrix} \]  \hspace{1cm} (35)

Assuming that \([L(t)]\) is a solution matrix of equation (34) with initial condition \([L(0)] = [I]\) then the transition matrix of equation (34) is defined by \([L(T)]\) with \(T = 2\pi/\omega_1\). For a flexible rotor system, the derivatives of nonlinear generalized forces shown in the last term of equation (3b) can be obtained by the same way for each squeeze-film damper. With knowing the transition matrix, the stability of the system motion can be determined.

**NUMERICAL EXAMPLES AND RESULTS**

**Example 1:**

The rotor system studied by Taylor and Kumar (1980) for the response analysis is employed to demonstrate the system stability. The system parameters are given by
- **rotor mass** \(m = 33.43 \text{ Kg}\)
- **bearing radius** \(R = 64.80 \text{ mm}\)
- **bearing width** \(L = 22.70 \text{ mm}\)
- **bearing clearance** \(C_r = 100 \mu\text{m}\)
- **viscosity of lubricant** \(\mu = 2.06 \times 10^{-3} \text{ N} \cdot \text{sec/m}^2\)
- **stiffness of centering spring** \(k = 2.154 \times 10^7 \text{ N/m}\)

The following cases of excitations are studied:
- **side load** \(F_s = 0.0, 300, 600, 900, 1200 \text{ N}\)
- **mass eccentricity** \(\delta = 0.125, 0.250, 0.375\)

It is assumed that the oil film is fully cavitated and a natural frequency defined as the stiffness of centering spring over the rotor mass is of the value \(\omega_m = 802.7 \text{ rad/sec}\).

For various mass eccentricity and side loads, the periodic response of the system can be obtained by individually choosing proper harmonic components. Figure 2 shows the nondimensional constant offset of the journal center in \(\omega\)-direction versus journal rotating speed. Figure 3 shows the nondimensional semi-major axis of the first harmonic response versus the journal rotating speed. Figures 4—6 show the nondimensional semi-major axis of the second harmonic response versus journal rotating speed for \(\delta = 0.125, 0.250\), and 0.375, respectively. The solid lines shown in Figures 2—6 represent the stable motion, i.e., \(|\lambda|_{\text{max}} < 1.0\) and the dashed lines represent the unstable motion, i.e., \(|\lambda|_{\text{max}} \geq 1.0\). The higher harmonic components of response are found very small compared to the first harmonic component of response and negligible.

The stability of synchronous response are shown in Figures 7—9 for \(\delta = 0.125, 0.250\), and 0.375, respectively. For the case of no side load, i.e., \(F_s = 0\), the steady state response is centric circular synchronous motion. The stability has been studied by Taylor and Kumar (1980) and Greenhill and Nelson (1982). It can be shown that, for small eccentricity (e.g., \(\delta = 0.125\)), the circular synchronous motion is stable and unique. However, if the eccentricity is increased, different dynamic phenomena will occur. For the case of \(\delta = 0.250\), there exist the jump phenomena for the circular synchronous motion when the rotating speed pass through \(\Omega \approx 1250.0 \text{ rad/sec}\). If the eccentricity is continuously increased (i.e., \(\delta = 0.375\)), it is found that the jump phenomena occur at \(\Omega \approx 1500.0 \text{ rad/sec}\) and a bistable operation behavior appear as \(\Omega > 1500.0 \text{ rad/sec}\). The bistable operation behavior and jump phenomena have been shown very important (Gunter et al., 1977; Botinan and Samaha, 1982; Rabinowitz and Hahn, 1983). The existence of side load will distort the orbits of circular synchronous motion and result in subharmonic whirl motion. In addition, it can destabilize the...
Consider the case of $\delta = 0.250$ and $F_s = 900N$. It is shown that if the journal rotating speed is in the range of 1900 rad/sec—2200 rad/sec, there exist no stable synchronous motion. Using the numerical method proposed by Hwang and Shiau (1991) for solving the subharmonic whirl motions, one will obtain $\frac{1}{2}$-subharmonic whirl motion with a period of $T_\frac{1}{2} = 2T_1 = 2(\frac{2\pi}{N})$. A direct numerical integration of the equations of motion has also been applied to compare the results. Figures 10a and 10b show the transient behaviors of subharmonic whirl motion calculated by the integration with different initial conditions for the time history from $1T_1$ to $14T_1$ with $\Omega = 2000$ rad/sec. Figure 10c shows the orbit of the consecutive five periods of time history from $15T_1$ to $19T_1$ for the case of Figure 10a and it converges to a limit double-loop motion which is very consistent with the solution shown in Figure 10d by the trigonometric collocation method (TCM). The stability analysis shows $|A|_{max} = 0.750$. However, when the journal is rotating at a speed $\Omega = 2200$ rad/sec which is near the boundary between $\frac{1}{2}$-subharmonic whirl motion and synchronous motion, the behaviors will be significantly changed. Figures 11a and 11b show the transient behaviors of subharmonic whirl motion obtained by the integration with different initial conditions for the time history from $17T_1$ to $14T_1$. Figure 11c shows the consecutive five periods of time history from $15T_1$ to $19T_1$ for the case of Figure 11a. The steady state solution shown in Figure 11d predicted by the TCM is of a maximum absolute eigenvalues of transition matrix, $|A|_{max} = 1.029$, which implies that the predicted steady state $\frac{1}{2}$-subharmonic whirl motion is unstable. For the case of $\Omega = 2300$ rad/sec, Figure 12a shows the transient behavior for the time history from $1T_1$ to $28T_1$. Figure 12b shows the transient behaviors with time history after $28T_1$, and Figure 12c shows the final solution. The final solution by integration is of very good agreement with the predicted solution by the TCM. It is stable because of $|A|_{max} = 0.819$.

It is expected that the ultra subharmonic whirl motion (i.e. $\frac{3}{2}$-subharmonic, $\frac{1}{2}$-subharmonic, etc.) will appear in the high rotating speed. The way of the determination of high order subharmonic whirl motion is the same as that for $\frac{1}{2}$-subharmonic whirl motion or synchronous motion case except that $\omega_1$ is equal to $\Omega/n$, $n = 3, 4, \ldots$.

Example 2:

The example considers a flexible rotor system (Hwang and Shiau, 1991) shown in Figure 13 which consists of rigid discs mounted at stations 1, 4, 5, and 12, and three linear isotropic supports located at stations 3, 6, and 13. Two squeeze-film dampers with centering spring are set at stations 3 and 13. The data of shaft and corresponding material properties are given in Table 1. The mass properties of rigid discs are listed in Table 2. The stiffness properties of the linear isotropic supports and the data of squeeze-film dampers are indicated in Table 3. The rigid disk at station 13 is of mass eccentricity 20.32 $\mu$m. In addition, the system is assumed to be horizontal so that the gravitational effect is considered as a side load distribution.

Using the modeling approach of Generalized Polynomial Expansion Method (GPEM) and introducing the idea of component mode synthesis, a set of nonlinear differential equations (Hwang and Shiau, 1991) which govern the system motion, can be obtained. Moreover, the hybrid numerical method, which has the merits of both HBM and TCM, is used to solve for the nonlinear response. Figures 14 and 15 show the nondimensional semi-major axis versus rotating speed for the first and the second harmonic motions associated with a constant offset relative to eccentricity for stations 3 and 13, respectively. It is also found that the magnitude of nondimensional semi-major axis for the third, the fourth, and higher harmonic motion are smaller than $10^{-3}$. Since the steady state response is nearly a circular synchronous motion, the nonlinear squeeze-film forces can be approximated by using equation (8). It is indicated that the steady state response using equation (8) of dashed lines shown in Figures 14 and 15, are in good agreement with that using equation (6) which is the model of non-circular motion. The maximum absolute eigenvalues of the transition matrix $|L(7)|$ are shown in Figure 16. It indicates that the nonlinear steady state response obtained are stable equilibrium.

**DISCUSSION AND CONCLUSION**

To investigate the dynamic characteristic of a complicate nonlinear system, the direct numerical integration method may be the simplest way. However, it may be inefficient for solving the steady state response. The methods such as TCM, HBM, and the hybrid numerical method described by Hwang and Shiau (1991) are of better efficiency to predict the steady state solution. The stability of steady state synchronous or subharmonic whirl motion are examined. The derivatives of the nonlinear hydrodynamic forces of squeeze-film damper with respect to the displacement and velocity of journal center are expressed in closed forms, which are required in the stability analysis.

For the application of methods of TCM, HBM, and the hybrid numerical method, one needs an initial guess of motion for the successive iterations to obtain the steady state response. If the solution of synchronous motion is assumed, the predicted solution can be found without difficulty. However, if the solution of subharmonic whirl motion is assumed, it requires more...
computational effort to obtain the predicted solution. It is noted that the motion usually converges first to the response of synchronous motion. To obtain the solution of subharmonic whirl motion, it is suggested to carefully choose the initial value or modify the iterative process. The transition matrix which governs the stability of rotor bearing system can be determined numerically with desirable accuracy. If the efficiency is concerned, the analysis with component modal truncation may be a better choice.

It should be noted that the superharmonic components of a periodic solution are also considered in the response analysis for both synchronous motion and subharmonic whirl motion, except for the centric circular synchronous motion. However, they are small compared to the component of fundamental frequency and negligible. Furthermore, the results conclude that large unidirectional side load may cause unstable synchronous equilibrium response when the journal is of rotating speed twice the synchronous resonant speed. In this case, the $\frac{1}{2}$-subharmonic whirl motion will appear. For small unidirectional side load, the transient behavior will also be a $\frac{1}{2}$-subharmonic whirl motion when the journal is rotating at twice the synchronous resonant speed. However, the steady state response is synchronous.

REFERENCES


Table 1 Date of the shaft of Example 2

<table>
<thead>
<tr>
<th>Station No.</th>
<th>Axial Distance to Station (cm)</th>
<th>Inner Radius (cm)</th>
<th>Outer Radius (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4.29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>8.89</td>
<td>1.42</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10.49</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>20.17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>27.69</td>
<td>1.96</td>
<td>2.05</td>
</tr>
<tr>
<td>7</td>
<td>44.70</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>59.44</td>
<td>2.69</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>74.68</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>89.92</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>105.16</td>
<td>2.26</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>120.14</td>
<td>1.42</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>127.94</td>
<td>2.21</td>
<td></td>
</tr>
</tbody>
</table>

\[ E = 20.69 \times 10^{10} \text{ N/m}^2; \quad \rho = 8193.0 \text{ Kg/m}^3 \]

Table 2 Mass properties of fixed rigid disc of Example 2

<table>
<thead>
<tr>
<th>Station No.</th>
<th>Mass (kg)</th>
<th>Polar Inertia (kg cm² × 10⁻⁶)</th>
<th>Transverse Inertia (kg cm² × 10⁻⁶)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11.38</td>
<td>19.53</td>
<td>9.82</td>
</tr>
<tr>
<td>4</td>
<td>7.88</td>
<td>16.70</td>
<td>8.35</td>
</tr>
<tr>
<td>8</td>
<td>7.70</td>
<td>17.61</td>
<td>8.60</td>
</tr>
<tr>
<td>12</td>
<td>21.70</td>
<td>44.48</td>
<td>22.24</td>
</tr>
</tbody>
</table>

Fig. 1 Model of rigid rotor carried in squeeze-film damper

Fig. 2 Nondimensional constant offset in \( \nu \)-direction versus rotating speed of Example 1
Fig. 3 Nondimensional semi-major axis of the first harmonic response versus rotating speed of Example 1

Fig. 4 Nondimensional semi-major axis of the second harmonic response versus rotating speed of Example 1 with \( \delta = 0.125 \)

Fig. 5 Nondimensional semi-major axis of the second harmonic response versus rotating speed of Example 1 with \( \delta = 0.250 \)

Fig. 6 Nondimensional semi-major axis of the second harmonic response versus rotating speed of Example 1 with \( \delta = 0.375 \)
Fig. 7 Stability boundary of the response of Example 1 for \( \delta = 0.125 \)

Fig. 8 Stability boundary of the response of Example 1 for \( \delta = 0.250 \)

Fig. 9 Stability boundary of the response of Example 1 for \( \delta = 0.375 \)

Fig. 10 Orbits of \( \frac{1}{2} \)-subharmonic whirl motion of Example 1 for \( \delta = 0.250 \), \( F_s = 900 \) N, and \( \omega = 2000.0 \) rad/sec
### Table 3 Data of isotropic supports and squeeze-film dampers of example 2

<table>
<thead>
<tr>
<th>Station No.</th>
<th>Stiffness of Center Spring (N/m)</th>
<th>Damping of Center Spring (N·sec/m)</th>
<th>Radius of the Journal (mm)</th>
<th>Axial Width of Bearing (mm)</th>
<th>Radius Clearance of Bearing (μm)</th>
<th>Viscosity of Lubricant (N·sec/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$0.44 \times 10^7$</td>
<td>0.0</td>
<td>50.8</td>
<td>25.4</td>
<td>152.4</td>
<td>$1.5 \times 10^{-2}$</td>
</tr>
<tr>
<td>6</td>
<td>$5.72 \times 10^7$</td>
<td>0.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>13</td>
<td>$1.30 \times 10^7$</td>
<td>0.0</td>
<td>50.8</td>
<td>25.4</td>
<td>152.4</td>
<td>$1.5 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

**Fig. 11** Orbits of $\frac{1}{2}$-subharmonic whirl motion of Example 1 for $\delta = 0.250$, $F_s = 900$ N, and $\omega = 2200.0$ rad/sec

**Fig. 12** Orbits of $\frac{3}{2}$-subharmonic whirl motion of Example 1 for $\delta = 0.250$, $F_s = 900$ N, and $\omega = 2300.0$ rad/sec
**Fig. 13** Rotor model of Example 2

**Fig. 14** Nondimensional semi-major axis versus rotating speed for the first and second harmonic at station 3 of Example 2

**Fig. 15** Nondimensional semi-major axis versus rotating speed for the first and second harmonic at station 13 of Example 2

**Fig. 16** Maximum absolute eigenvalues of Example 2