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## Investigation of the Vibration of a Blade with Friction Damper by HBM

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### ABSTRACT

The friction damper has been widely used to reduce the resonant vibration of blades. The most commonly used methods for studying the dynamic behavior of a blade with a friction damper are direct integration methods. Although the harmonic balance method (HBM) is a well known method for studying nonlinear vibration problems, generally only a one-term approximation has been proposed to study the nonlinear vibration of a frictionally damped blade. In this work, a procedure of the HBM with a multi-term approximation is proposed. The results show that the steady state response and other related behavior of a frictionally damped blade can be predicted accurately and quickly by the HBM with a multi-term approximation.

### INTRODUCTION

The friction damper has been widely used to reduce the resonant vibration of blades. In the past, many works have been done to investigate the dynamic behavior of a blade with a friction damper either by the macroslip approach (Griffin, 1980; Srinivasan, 1986; Wang, 1990) or by the microslip approach (Meng, et al., 1986a, b). Generally, the steady state nonlinear forced vibration of a blade was investigated by direct time-step integration with iterative process. It is very time consuming because the decay of the transient vibration generally is very slow. Although the harmonic balance method (HBM) is a well known method to study the nonlinear vibration problems, usually only the HBM with one-term harmonic was used to investigate the harmonic forced vibration of a blade with a friction damper (Meng and Griffin, 1985). This is due to the difficulty that the nonlinear friction force is dependent on the stepwise stick-slip motion which is not known in advance. As a consequence, the number of simultaneous algebraic equations found from the HBM is less than the number of unknowns if the response is assumed to consist of many harmonic terms. The slowly varying parameter method has also been used to study the harmonic response of a blade with a friction damper (Sinha and Griffin, 1984). However, only one term of harmonic has been considered in that work. A modified HBM with multi-harmonic approximation has been proposed by Pierre (1985) to investigate a dry friction damped system. The method is also known as incremental harmonic balance (IHB).

However, the IHB method is no more efficient than direct numerical integration when the number of harmonics is increased. Besides the IHB, an alternating frequency/time (AFT) method (Cameron and Griffin, 1989) can also accommodate multiple harmonics for the study of dry friction problem. However, the method has to switch the calculation alternately in frequency and time domains. Besides, the aliasing and leakage problems also must be handled carefully.

The traditional HBM due to its simplicity has been widely accepted in the past. However, as mentioned, it is difficult to apply the traditional HBM with multiple harmonics to investigate the vibration of a dry friction damped blade. In this work, a method was developed to overcome the difficulty when the HBM with many harmonic terms was used. Some extra constraint equations were found to supplement the algebraic equations found from the traditional harmonic balance method. The accuracy and the computational time from the multi-term HBM were compared with the results from the direct time integration.

### ANALYSIS

As mentioned, two different friction models, i.e., macroslip and microslip approaches, have been proposed in the past. Generally speaking, the microslip approach can provide more accurate results only when the normal preload of the damper is high, and at the expense of higher computational effort, as discussed by Meng (1986a) and Wang (1991). Therefore, the macroslip approach was adopted in this work.

A typical single blade with a massless, flexible, blade-to-ground friction damper is shown in Fig.1. If the blade is discretized by the finite element method, then the equation of motion of the blade can be written as:

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F\} + \{F_n\} \quad (1)$$

where  $\{X\}$  is the generalized displacement vector,  $[M]$ ,  $[C]$  and  $[K]$  are the mass, damping and stiffness matrices;  $\{F\}$  and  $\{F_n\}$  are the linear external force and the non-linear friction force vectors. If only a one-mode approximation is used to investigate the response of the blade near a resonance frequency, then the blade of Figure 1 can be approximated by a single degree of freedom system, as shown in Fig.2. The equation of motion of

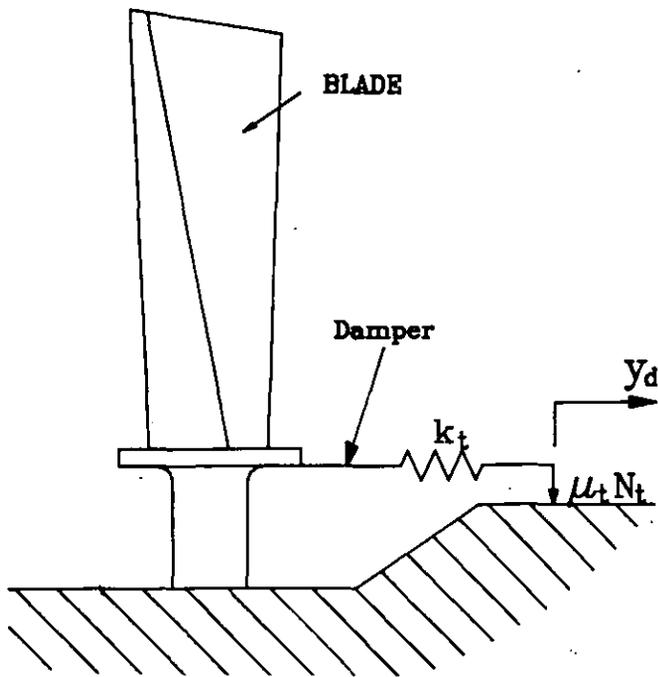


Fig. 1 Schematic of a single blade with a friction damper

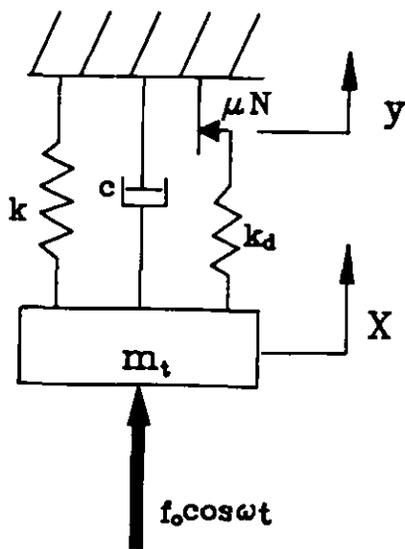


Fig. 2 Model of one-degree-of-freedom system

the single-mode model can be written as

$$m\ddot{x} + c\dot{x} + kx = f - f_n \quad (2)$$

where  $f$  is the external excitation force and  $f_n$  is the friction force due to the friction damper.  $N$ ,  $\mu$  and  $k_d$  in Fig. 2 represent the normal preload, the friction coefficient of the friction damper and the stiffness of the damper in the direction of relative

motion, respectively. According to the macroslip approach, the friction force,  $f_n$ , can be expressed as

$$f_n = \begin{cases} +k_d(x-y) & \text{when } k_d|x-y| \leq \mu N \\ +\mu N \text{sign}(\dot{y}) & \text{when } k_d|x-y| \geq \mu N \end{cases} \quad (3)$$

where  $y$  is the displacement of the friction damper, as shown in Fig. 2. If the external excitation force is a simple harmonic force  $f = f_0 \cos \omega t$ , the steady state solution of equation (2) can then be found by the harmonic balance method (HBM). In the following section, the one-term, two-term and multi-term approximations will be discussed separately.

#### One-Term Approximation

According to the HBM the steady state solution of equation (2) can be expressed as a Fourier expansion,

$$x = \sum_{i=1}^{\infty} [A_i \cos(i\omega t) + B_i \sin(i\omega t)] \quad (4)$$

In this section, only a one-term approximation will be used, i.e.,

$$\begin{aligned} x &= A_1 \cos \omega t + B_1 \sin \omega t \\ &= A_1 \cos \theta + B_1 \sin \theta \equiv R \cos \theta \end{aligned} \quad (5)$$

where  $\theta = \omega t$  and  $\theta = \omega t - \phi$ . The angle  $\phi$  is the phase difference between the external excitation force and the response. According to equation (5) the angle  $\theta$  should be equal to zero when the response is maximum,  $R$ . A typical relationship between the response and the friction force is shown in Fig. 3. The damper is in stick and slip conditions during the AB and BC sections respectively. Then the velocity of the damper,  $\dot{y}$  ( $\dot{y}$  is indicated in Fig. 2) should be as:

$$\dot{y} = \begin{cases} 0 & \text{when } 0 \leq \theta \leq \theta^* \\ \dot{x} & \text{when } \theta^* \leq \theta \leq \pi \end{cases} \quad (6)$$

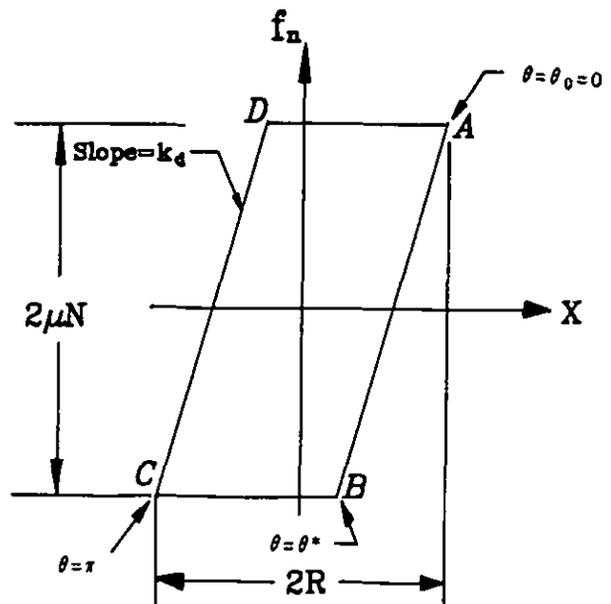


Fig. 3 Frictional damping loop, friction force vs the response (one-term approximation)

From equation (6), one has

$$y = \begin{cases} c_1 & \text{when } 0 \leq \theta \leq \theta^* \\ x+c_2 & \text{when } \theta^* \leq \theta \leq \pi \end{cases} \quad (7)$$

The two constants  $c_1$  and  $c_2$  can be determined by the analysis in what follows. During the stick section, the friction force is equal to zero when  $x$  is equal to  $y$ . According to Fig. 3,  $f_n = 0$  (the middle point of line AB), when  $x$  is equal to  $R - \mu N/k_d$ . In other words, one finds that  $c_1 = R - \mu N/k_d$ . During the slip section, the difference between  $x$  and  $y$  is hold constant, i.e.,  $c_2$  which is the horizontal distance between the point B and the middle point of line AB in Fig. 3. Then one can find that  $c_2 = \mu N/k_d$ . From equation (3), the friction force can then be expressed as:

$$f_n = k_d(x-y) = k_d(x-R) + \mu N, \text{ when } 0 \leq \theta \leq \theta^* \quad (8a)$$

$$f_n = \mu N \text{sign}(\dot{y}) = \mu N \text{sign}(\dot{x}) = -\mu N, \text{ when } \theta^* \leq \theta \leq \pi \quad (8b)$$

Equation (8a) should be equal to equation (8b) when  $\theta = \theta^*$ , then one has:

$$\theta^* = \cos^{-1}[1 - 2\mu N/(k_d R)] \quad (9)$$

Because the response is assumed to be a simple harmonic form, the response cycle can be divided into two symmetric half cycles, i.e., the half cycle ABC is equal to the other half cycle CDA in Fig. 3. So, one can find that

$$f_n(\theta + \pi) = -f_n(\theta), \quad 0 \leq \theta \leq \pi \quad (10)$$

With equations (8) and (10), the friction force can be expanded by Fourier series, and only the first harmonic term is kept,

$$f_n(\theta) = F_c \cos \theta + F_s \sin \theta \quad (11)$$

$$\text{with } \begin{cases} F_c = (-k_d R / \pi) [\theta^* - 0.5 \sin(2\theta^*)] \\ F_s = (-4\mu N / \pi) [1 - \mu N / (k_d R)] \end{cases}$$

Substitute equations (5) and (11) into equation (2) and separate the coefficients of  $\cos \theta$  and  $\sin \theta$ , one has:

$$(k - m\omega^2)R + F_c = f_0 \cos \phi \quad (12a)$$

$$c\omega R - F_s = f_0 \sin \phi \quad (12b)$$

or

$$[(k - m\omega^2)R + F_c]^2 + [c\omega R - F_s]^2 = f_0^2 \quad (12c)$$

$$\phi = \sin^{-1}[(c\omega R - F_s)/f_0] \quad (12d)$$

Equations (12c) and (12d) are nonlinear equations with two unknowns  $R$  and  $\phi$  which can be solved by traditional Newton-Raphson method. In practice, only the unknown  $R$  needs to be found by iteration, the angle  $\phi$  can be obtained directly from equation (12d) providing the  $R$  is known. The response is then known from equation (5). The one term approximation is not the target of the present work, however, it may provide valuable information for multi-term approximation.

### Two-Term Approximation

From the result of the direct time integration, it is known that the steady state response of the system in Fig. 2 contains only the odd harmonic components of the external excitation force. Therefore, if a two-term approximation is assumed, according to equation (4), the response can be written as

$$\begin{aligned} x &= A_1 \cos \omega t + B_1 \sin \omega t + A_3 \cos 3\omega t + B_3 \sin 3\omega t \\ &= A_1 \cos \theta + B_1 \sin \theta + A_3 \cos 3\theta + B_3 \sin 3\theta \end{aligned} \quad (13)$$

Let the maximum value of  $x$  be denoted by  $A_m$  and the corresponding  $\theta$  be denoted by  $\theta_0$ , as shown in Fig. 4. The point B in Fig. 4 is the slip impending point, and the value of  $\theta$  at this point is indicated by  $\theta^*$ . In a manner similar to the one-term approximation, the friction force during the stick portion (line AB) can be expressed as

$$\begin{aligned} f_n &= k_d(x - A_m) + \mu N \\ &= k_d[A_1(\cos \theta - \cos \theta_0) + B_1(\sin \theta - \sin \theta_0) + A_3(\cos 3\theta - \cos 3\theta_0) \\ &\quad + B_3(\sin 3\theta - \sin 3\theta_0)] + \mu N, \text{ when } \theta_0 \leq \theta \leq \theta^* \end{aligned} \quad (14)$$

During the slip portion, the friction force is

$$f_n = -\mu N, \text{ when } \theta^* \leq \theta \leq \pi + \theta_0 \quad (15)$$

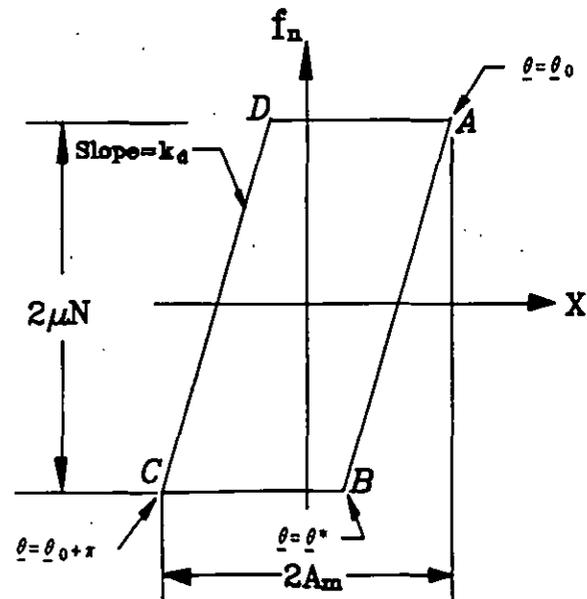


Fig. 4 Frictional damping loop, multi-term approximation

So far,  $\theta_0$  and  $\theta^*$  are unknown. This is the main difference between the one-term and the multi-term approximations. How to determine these unknowns will be discussed later. With equations (14) and (15), the friction force can be expanded as a Fourier series, however, only the first two terms are kept,

$$f_n = F_{c1} \cos \theta + F_{s1} \sin \theta + F_{c3} \cos 3\theta + F_{s3} \sin 3\theta \quad (16)$$

The coefficients  $F_{c1}$ ,  $F_{s1}$ ,  $F_{c3}$  and  $F_{s3}$  can be found in the Appendix. By substituting equations (13) and (16) into equation (2), and setting the coefficients of  $\cos \theta$ ,  $\sin \theta$ ,  $\cos 3\theta$ , and  $\sin 3\theta$  equal to zero, one has,

$$(k - m\omega^2)A_1 + \omega c B_1 - f_0 + F_{c1} = 0 \quad (17)$$

$$(k - 9m\omega^2)A_3 + 3\omega c B_3 + F_{c3} = 0 \quad (18)$$

$$(k - m\omega^2)B_1 - \omega c A_1 + F_{s1} = 0 \quad (19)$$

$$(k - 9m\omega^2)B_3 - 3\omega c A_3 + F_{s3} = 0 \quad (20)$$

Because the  $F_{c1}$ ,  $F_{s1}$ ,  $F_{c3}$  and  $F_{s3}$  are function of  $\underline{\theta}_0$  and  $\underline{\theta}^*$  which are unknown, the four equations (17)–(20) have six unknowns, i.e.,  $A_1$ ,  $B_1$ ,  $A_3$ ,  $B_3$ ,  $\underline{\theta}_0$  and  $\underline{\theta}^*$ . So, one must find other equations to supplement equations (17)–(20). The response  $x$  is maximum (with value  $A_m$ ) when  $\underline{\theta} = \underline{\theta}_0$ . Thus, the following equations should be satisfied:

$$\left. \frac{\partial x}{\partial t} \right|_{\underline{\theta}=\underline{\theta}_0} = -A_1 \sin \underline{\theta}_0 + B_1 \cos \underline{\theta}_0 - 3A_3 \sin 3\underline{\theta}_0 + 3B_3 \cos 3\underline{\theta}_0 = 0 \quad (21)$$

$$\left. \frac{\partial^2 x}{\partial t^2} \right|_{\underline{\theta}=\underline{\theta}_0} = -A_1 \cos \underline{\theta}_0 - B_1 \sin \underline{\theta}_0 - 9A_3 \cos 3\underline{\theta}_0 - 9B_3 \sin 3\underline{\theta}_0 < 0 \quad (22)$$

It is also known that the friction forces in equations (14) and (15) should be equal when  $\underline{\theta} = \underline{\theta}^*$ . Thus,

$$k_d [A_1 (\cos \underline{\theta}^* - \cos \underline{\theta}_0) + B_1 (\sin \underline{\theta}^* - \sin \underline{\theta}_0) + A_3 (\cos 3\underline{\theta}^* - \cos 3\underline{\theta}_0) + B_3 (\sin 3\underline{\theta}^* - \sin 3\underline{\theta}_0)] + \mu N = -\mu N \quad (23)$$

or in a compact functional form.

$$G(\underline{\theta}^*, \underline{\theta}_0) = 0 \quad (24)$$

If  $\underline{\theta}_0$  and  $\underline{\theta}^*$  are known, then equations (17)–(20) are linear equations with  $A_1$ ,  $A_3$ ,  $B_1$ ,  $B_3$  as unknowns which can be solved easily. The values of  $\underline{\theta}_0$  and  $\underline{\theta}^*$  can be determined by equations (21)–(23) which are nonlinear. In summary, equations (17)–(23) can be used to determine the unknowns  $A_1$ ,  $A_3$ ,  $B_1$ ,  $B_3$ ,  $\underline{\theta}_0$ ,  $\underline{\theta}^*$ .

The procedure is summarized as follows:

- (1) Give proper initial guess values of  $\underline{\theta}_0$  and  $\underline{\theta}^*$ , and then  $F_{c1}$ ,  $F_{c3}$ ,  $F_{s1}$  and  $F_{s3}$  can be found (see Appendix).
- (2) The unknowns  $A_1$ ,  $A_3$ ,  $B_1$ ,  $B_3$  can be determined from the linear equations (17)–(20) by direct matrix inversion.
- (3) With  $A_1$ ,  $A_3$ ,  $B_1$ , and  $B_3$ , the condition of equation (22) is then checked:

- (a) If equation (22) is satisfied, then go to step (4)
- (b) If equation (22) is not satisfied, then the initial guess values of  $\underline{\theta}_0$  and  $\underline{\theta}^*$  are modified as: new  $\underline{\theta}_0 = \text{old } \underline{\theta}_0 + \pi$ , new  $\underline{\theta}^* = \text{old } \underline{\theta}^* + \pi$ . After the modification, then go back to step (1).

- (4) With  $A_1$ ,  $A_3$ ,  $B_1$ , and  $B_3$ , equations (21) and (23) are then used to find the exact values of  $\underline{\theta}_0$  and  $\underline{\theta}^*$  by the Newton–Raphson iteration method. The flow chart of the procedure is given in Fig. 5.

The modification in step (3) can be understood easily because, according to Fig. 4, the response is symmetric, i.e.,  $x(\underline{\theta}) = -x(\underline{\theta} + \pi)$ .

It is known that the most important step in using the Newton–Raphson iteration method is the guess of the initial values. How to guess proper initial values of  $\underline{\theta}_0$  and  $\underline{\theta}^*$  will be discussed in the next section.

The above derivation and solution procedure can easily be extended to the multi-term approximation. If an  $n$ -term approximation is used, then one can find  $2n$  linear equations like equations (17)–(20) and two nonlinear equations like equations (21) and (23).

## RESULTS AND DISCUSSION

In this section, three main topics will be discussed, i.e., the accuracy of the HBM, the computation time, and the effect of the initial guess values. Before the discussion, the system parameters of Fig. 2 are given here:  $m=1.24$ ,  $k=17890$ ,  $c=3$ , and  $k_d=3500$ .

### Accuracy of the HBM

As mentioned, generally only one-term HBM was used to investigate the vibration of a blade with a friction damper. In some cases, for instance, in fully stuck or fully slipping conditions, the one-term approximation is accurate enough to predict the response. However, the one-term approximation generally may cause significant error. Here, an example is given to show this situation. Fig. 6 shows a response found by the direct time integration, one-term HBM, two-term HBM and three-term HBM respectively. The vertical axis is the normalized response  $A/A_m$ , where  $A_m$  is the maximum amplitude found by the direct time integration. In this case the excitation frequency  $\omega$  is set to be  $1/3 \omega_0$ , where  $\omega_0$  is the natural frequency of the system of Fig. 2 without the friction damper. It is known that if the excitation frequency  $\omega$  and the amplitude of excitation force  $f_0$  are determined, then an optimal normal load  $N$  which results in minimum response can be found. The normalized normal load  $\mu N/f_0$  in Fig. 6 is set to the optimal value  $\mu N/f_0 = 0.145$ . One can find that the one-term approximation results in significant error. In this case, the three-term approximation can accurately predict the response, and its result is difficult to be distinguished from the result of the direct integration in the Figure. The corresponding frictional damping loop is shown in Fig. 7. One can find that the frictional damping can not be simulated accurately by the one-term approximation. The above result indicates that only the multi-term HBM can accurately predict the response of a blade with a friction damper. The result also demonstrates that the multi-term HBM proposed in this work is feasible. Theoretically, the HBM with more terms will result in more accurate solution. In practice, according to our experience, it needs no more than three terms. The question of the computational time will be discussed in the next section.

### Computer Time

No matter how many terms, say  $n$  terms, are kept in the HBM, only two nonlinear equations (i.e., equations (21) and (23)) and  $2n$  simultaneous linear equations (i.e., equations (17)–(20)) are needed to find all the unknowns. As shown in Fig. 5, the two nonlinear equations are solved by the iteration method, and the  $2n$  linear equations must be solved once for each iteration. In other words, if  $m$  iterations are needed until convergence is reached, then the  $2n$  linear equations must be solved  $m$  times. Therefore, how to reduce the number of iteration is very important. It is known that the initial guess values are very important for the iteration process. An improper initial guess value may drastically increase the number of iterations or cause divergence. Thus, to discuss the computational time, the effect of the number of terms should be considered together with the effect of the initial guess values. First, the effect of the initial guess values will be discussed.

The nonlinearity of the vibratory blade comes only from the nonlinear friction force. It is known that the friction force can be expanded by Fourier series with the fundamental frequency equal to the external excitation frequency  $\omega$ . In other words, the vibration of the blade is mainly caused by a harmonic force with a frequency equal to  $\omega$ . Thus, the solution obtained from the one-term approximation should be a good initial guess value for the multi-term approximation. For multi-term approximation, one should give the initial guess values of  $\underline{\theta}_0$  and  $\underline{\theta}^*$ . In this work, it is suggested that the  $\underline{\theta}_0$  and  $\underline{\theta}^*$  obtained from the one-term approximation can be used as the initial guess values of  $\underline{\theta}_0$  and  $\underline{\theta}^*$  for the multi-term approximation. According

to equation(5), one has the relations:  $\underline{\theta}_0 = \theta_0 + \phi$  and  $\underline{\theta}^* = \theta^* + \phi$ . As pointed out in the one-term approximation,  $\theta_0$  is zero, while  $\theta^*$  and  $\phi$  can be obtained from equation (9) and equation (12) respectively. To understand the effect of the initial guess values on the computer time, different guess values are assumed and the necessary computer time is recorded. The result is shown in Fig. 8. The horizontal axis "shift of initial value" means the amount the initial guess values of  $\underline{\theta}_0$  and  $\underline{\theta}^*$  are shifted from the suggested initial values, i.e.,  $\underline{\theta}_0 = \phi$  and  $\underline{\theta}^* = \theta^* + \phi$ . For instance, "-0.2" indicates that initial guess values are given as:  $\underline{\theta}_0 = (1-0.2)\phi$ ,  $\underline{\theta}^* = (1-0.2)(\theta^* + \phi)$ . One can find that if the shift of the initial value is within -0.5 and 0.3, one can find a converged solution corresponding to the solution found from the direct integration. If the initial guess values are beyond this range, one may find another converged solution or the solution is diverged. Here "another converged solution" means a solution which is different from the solution found from the

direct time integration. So far, it is not sure that this "another solution" exists in practice. The result of Fig. 8 indicates that the  $\theta^*$  and  $\phi$  found from one-term approximation are the proper initial guess values of  $\underline{\theta}_0$  and  $\underline{\theta}^*$  for the multi-term approximation.

If the initial guess values are selected properly, the ratios of the computer time are 1:2.4:5.5:61 for one-term, two-term, three-term approximations and the direct time integration respectively. Note that the computer time for the two-term and three-term approximations includes the portion to find the proper initial guess values, i.e.,  $\theta^*$  and  $\phi$ . As mentioned, the HBM requires no more than three terms in order to obtain an accurate result. That is to say the HBM needs a computer time about one order of magnitude smaller than that of the direct integration. The above result is concluded not only from the above example, but also from many other calculated examples.

The advantage of the proposed HBM is especially significant when the response must be recalculated many times, for instance, to find the optimal normal load or the frequency

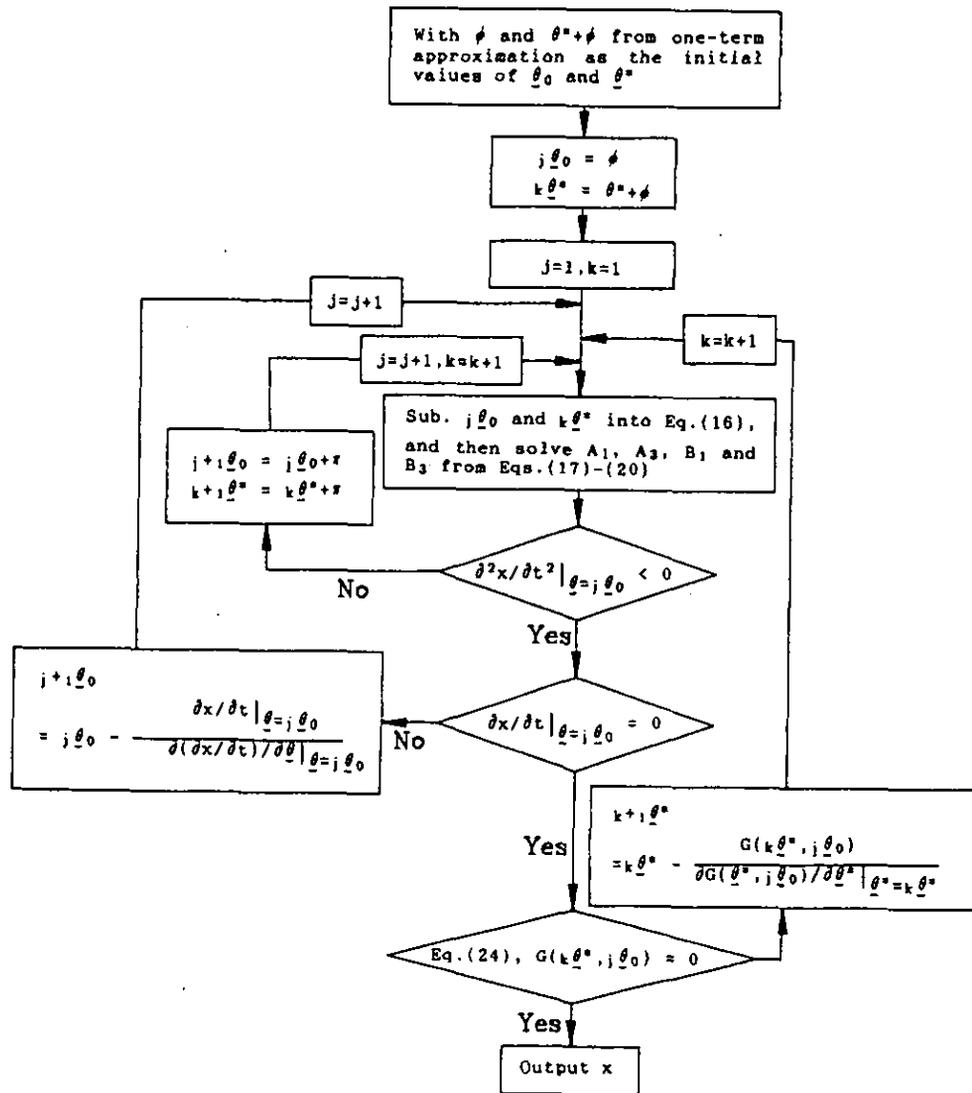


Fig. 5 Flow chart of the proposed iteration procedure

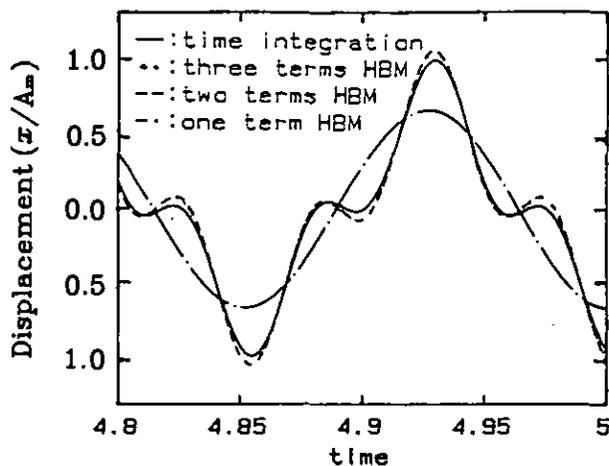


Fig. 6 Steady state forced response,  $w/w_0=0.334$ ,  $\mu N/f_0=0.145$

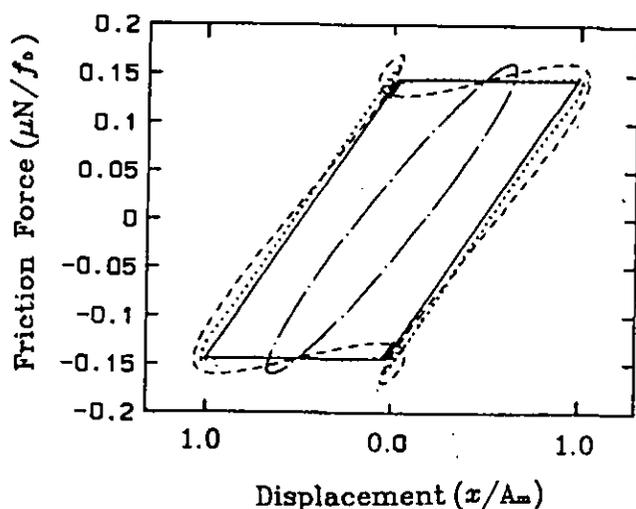


Fig. 7 Frictional damping loop, the legend is the same as Fig. 6

response curve. It is well known that there exists an optimal normal load  $N$  with which the blade will experience minimum forced vibration. The optimal normal load is an important design parameter of the friction damper. Generally, the optimal normal load can not be found directly except when the response is a pure simple harmonic function. The optimal normal load generally is found by observing the response for many different normal load values. In other words, a curve indicating the responses vs the normal load  $N$  should be created, and the minimum response corresponds to the optimal normal load. For each value of  $N$ , a new calculation must be executed. It is very time consuming if the direct integration method is used. For the proposed method, the answer can be found very quickly, because the solution for normal load  $N$  can be used as a very good initial guess value for finding the solution at  $N+\Delta N$ ,  $\Delta N$  is a small increment. Because the initial guess values are very close to the exact values, the iteration converges very quickly. The

frequency response curve, i.e., the response vs the excitation frequency, can also be found quickly by the proposed method. The response at a certain excitation frequency  $w$  can be used as a very good initial guess value for finding the response at frequency  $w+\Delta w$ .

In summary, there are two important features of the proposed method: (1) There are only two nonlinear equations no matter how many terms are kept in the HBM; (2) A proper initial guess value for the iteration generally can be found easily. Due to these features, the proposed method is very efficient for the steady state response and related analyses.

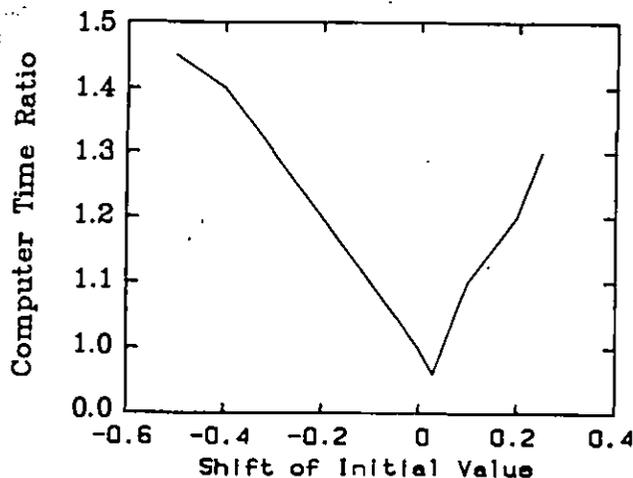


Fig. 8 The effect of initial guess values on the computer time

## CONCLUSION

The dynamic behavior of a blade with a friction damper has been intensively investigated in the past. Although the harmonic balance method (HBM) is a well known method, usually, usually only the HBM with a one-term (or one harmonic component) approximation was used to investigate the harmonic forced vibration of a blade with a friction damper. Generally, the HBM with an  $n$ -term approximation will result in  $2n$  simultaneous nonlinear algebraic equations in terms of  $2n$  unknown Fourier coefficients for a system with one-degree-of freedom. However, this is not the case in the proposed method. The main step of the proposed procedure is to find the impending time points (i.e.,  $\underline{\theta}_0$  and  $\underline{\theta}^*$ ) of stick and slip of the friction damper. Because the  $\underline{\theta}_0$  and  $\underline{\theta}^*$  are found from two nonlinear algebraic equations, the problem of the initial guess values of the  $\underline{\theta}_0$  and  $\underline{\theta}^*$  has also been discussed in this work. The results show that the HBM needs no more than three terms in order to obtain an accurate result. The main advantage of the HBM is the computational efficiency, it needs only about one-tenth of the computer time in comparison with the direct integration. Due to the drastic reduction of the computer time, it becomes possible to investigate many important characteristics of a frictionally damped blade in a very short time.

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#### APPENDIX

The coefficient of the Fourier series in equation (16) is derived in this appendix.

$$\begin{aligned}
 F_{c1} &= (2/\pi) \left\{ \int_{\theta_0}^{\theta^*} [k_d(x-A_m) + \mu N] \cos \theta d\theta - \int_{\theta^*}^{\theta_0 + \pi} \mu N \cos \theta d\theta \right\} \\
 &= (2/\pi) \left\{ \int_{\theta_0}^{\theta^*} k_d [A_1(\cos \theta - \cos \theta_0) + A_3(\cos 3\theta - \cos 3\theta_0) \right. \\
 &\quad \left. + B_1(\sin \theta - \sin \theta_0) + B_3(\sin 3\theta - \sin 3\theta_0)] \cos \theta d\theta \right. \\
 &\quad \left. + \int_{\theta_0}^{\theta^*} \mu N \cos \theta d\theta - \int_{\theta^*}^{\theta_0 + \pi} \mu N \cos \theta d\theta \right\} \\
 &= (2/\pi) \left\{ k_d [A_1 S_1 + A_3 S_2 + B_1 S_3 + B_3 S_4] + k_d [A_1 \cos \theta_0 \right. \\
 &\quad \left. + A_3 \cos 3\theta_0 + B_1 \sin \theta_0 + B_3 \sin 3\theta_0] (\sin \theta_0 \right. \\
 &\quad \left. - \sin \theta^*) + 2\mu N \sin \theta^* \right\}
 \end{aligned}$$

$$\begin{aligned}
 F_{c3} &= (2/\pi) \left\{ \int_{\theta_0}^{\theta^*} [k_d(x-A_m) + \mu N] \cos 3\theta d\theta - \int_{\theta^*}^{\theta_0 + \pi} \mu N \cos 3\theta d\theta \right\} \\
 &= (2/\pi) \left\{ k_d [A_1 S_2 + A_3 S_5 + B_1 S_6 + B_3 S_7] + (k_d/3) [A_1 \cos \theta_0 \right. \\
 &\quad \left. + A_3 \cos 3\theta_0 + B_1 \sin \theta_0 + B_3 \sin 3\theta_0] (\sin 3\theta_0 \right. \\
 &\quad \left. - \sin 3\theta^*) + (2\mu N/3) \sin 3\theta^* \right\}
 \end{aligned}$$

$$\begin{aligned}
 F_{s1} &= (2/\pi) \left\{ \int_{\theta_0}^{\theta^*} [k_d(x-A_m) + \mu N] \sin \theta d\theta - \int_{\theta^*}^{\theta_0 + \pi} \mu N \sin \theta d\theta \right\} \\
 &= (2/\pi) \left\{ k_d [A_1 S_3 + A_3 S_6 + B_1 S_8 + B_3 S_9] - k_d [A_1 \cos \theta_0 \right. \\
 &\quad \left. + A_3 \cos 3\theta_0 + B_1 \sin \theta_0 + B_3 \sin 3\theta_0] (\cos \theta_0 \right. \\
 &\quad \left. - \cos \theta^*) - 2\mu N \cos \theta^* \right\}
 \end{aligned}$$

$$\begin{aligned}
 F_{s3} &= (2/\pi) \left\{ \int_{\theta_0}^{\theta^*} [k_d(x-A_m) + \mu N] \sin 3\theta d\theta - \int_{\theta^*}^{\theta_0 + \pi} \mu N \sin 3\theta d\theta \right\} \\
 &= (2/\pi) \left\{ k_d [A_1 S_4 + A_3 S_7 + B_1 S_{10} + B_3 S_{11}] - (k_d/3) \right. \\
 &\quad \left. [A_1 \cos \theta_0 + A_3 \cos 3\theta_0 + B_1 \sin \theta_0 + B_3 \sin 3\theta_0] \right. \\
 &\quad \left. (\cos 3\theta_0 - \cos 3\theta^*) - (2\mu N/3) \cos 3\theta^* \right\}
 \end{aligned}$$

where

$$\begin{aligned}
 S_1 &= (\theta^* - \theta_0)/2 + (\sin 2\theta^* - \sin 2\theta_0)/4 \\
 S_2 &= (\sin 2\theta^* - \sin 2\theta_0)/4 + (\sin 4\theta^* - \sin 4\theta_0)/8 \\
 S_3 &= (\cos 2\theta_0 - \cos 2\theta^*)/4 \\
 S_4 &= (\cos 4\theta_0 - \cos 4\theta^*) + (\cos 2\theta^* - \cos 2\theta_0) \\
 S_5 &= (\theta^* - \theta_0)/2 + (\sin 6\theta^* - \sin 6\theta_0)/12 \\
 S_6 &= (\cos 4\theta_0 - \cos 4\theta^*) + (3/4)(\cos 2\theta^* - \cos 2\theta_0) \\
 S_7 &= (\cos 6\theta_0 - \cos 6\theta^*)/12 \\
 S_8 &= (\theta^* - \theta_0)/2 - (\sin 2\theta^* - \sin 2\theta_0)/4 \\
 S_9 &= (\sin 2\theta^* - \sin 2\theta_0)/4 - (\sin 4\theta^* - \sin 4\theta_0) \\
 S_{10} &= (\theta^* - \theta_0)/2 - (\sin 6\theta^* - \sin 6\theta_0)/12
 \end{aligned}$$