ABSTRACT

A two-step method is presented for the determination of reliable approximations of the probability density function of the forced response of a randomly mistuned bladed disk. Under the assumption of linearity, an integral representation of the probability density function of the blade amplitude is first derived. Then, deterministic perturbation techniques are employed to produce simple approximations of this function. The adequacy of the method is demonstrated by comparing several approximate solutions with simulation results.

INTRODUCTION

The accurate prediction of the dynamic behavior of turbomachines has been and still represents an important issue in the design of gas turbines. This continuous interest is in part due to the complexity of the fluid-structure interaction problem involved but also to the ever increasing sophistication of the components of the disk such as blade geometry, presence of shrouds, lacing wires, blade-to-blade or blade-to-ground dampers,... The difficulty of the analysis has naturally led to the formulation of simplified models based on physical assumptions. In particular, it has often been assumed that the blades were identical or equivalently that small differences in blade geometry and mechanical properties arising during the manufacturing process could be neglected. This hypothesis appears only logical at first view of the quality control inspections that are performed during the manufacturing process. However, such models have been unable to explain some blade failures, the so-called rogue blades.

In search for an explanation, some early investigators (Whitehead, 1966; Dye and Henry, 1969; Ewins, 1969) found that small variations in blade mechanical properties across the disk could result in a much larger increase in the forced vibration response of some blades. This lack of symmetry of the disk, traditionally referred to as mistuning, has since been investigated by quite a few researchers (Huang, 1982; Dugundji and Bundas, 1984; Ewins and Han, 1984; Kielb and Kaza, 1984; Basu and Griffin, 1986; Sinha, 1986; Griffin, 1988; Sinha and Chen, 1988, 1989; Wei and Pierre, 1988a, 1988b; Mignolet and Christensen, 1990 and references therein). In these investigations, it has been found that distributions of blade frequencies remaining within 5% or less of the design value can lead to amplitudes of vibration of certain blades that are as large as 2.43 times (Basu and Griffin, 1986) the tuned value, which is computed on the basis of identical blades. In fact, a theoretical study of this phenomenon (Whitehead, 1966) has shown that the increase in amplitude can be as large as \((1 + \frac{1}{N})/2\) where \(N\) denotes the number of blades. Clearly, the fatigue life of the high response blades will be substantially lower than the corresponding value predicted for the tuned system.

The influence of mistuning on the free vibration of the disk has also been investigated, especially in connection with the determination of the flutter boundary (Bendiksken, 1984; Dugundji and Bundas, 1984; Kaza and Kielb, 1984; Kielb and Kaza, 1984; Srinivasan and Fabunmi, 1984; Crawley and Hall, 1985; Kaza and Kielb, 1985; Nissim, 1985; Valero and Bendiksken, 1986 and references therein). Paradoxically, these studies have shown that mistuning has a beneficial effect on flutter; it pushes the appearance of this phenomenon to higher flow velocities. In particular, it was shown that a distribution of blades which alternatively have a high and a low natural frequency (alternate mistuning) is quite effective in raising the flutter velocity. From a practical point of view, this result is very interesting since such a mistuning is readily achieved by selecting the blades from two distinct groups of identical components. However, it should be noted that the corresponding disk consists of a tuned assembly of identical subsystem each composed of two adjacent blades. Thus, its forced vibration response is likely to be highly influenced by small variations in the mechanical properties of the subsystem which might occur during the manufacturing process.

The main difficulty encountered in the assessment of the effects of the unavoidable manufacturing mistuning lies in the lack of precise knowledge of the distribution of blade natural frequencies. In fact, it was recognized very early that the most appropriate model represents the blades' mechanical properties, mass and/or stiffness, as random variables whose means coincide with the design values. Further, their small standard deviations yield measures of quality of the manufacturing process. This characterization provides a well defined mathematical frame, namely, the theory of random vibrations, for the quantification of the effects of mistuning, in particular, for the determination of the minimum and mean values of the fatigue life. To perform this analysis, it is necessary to first obtain the probability density function of the amplitude of response of the bladed disk. This prior computation, which has already been considered by several researchers (Huang, 1982; Sinha, 1986; Sinha and Chen, 1988, 1989; Mignolet and Christensen, 1990, Wei and Pierre, 1990), represents the object of the
THE MODEL AND ITS DETERMINISTIC RESPONSE

Neglecting the effects of the nonlinearities associated with the structure, the fluid and their interaction, it is found that a bladed disk can be modeled as a linear multi-degree-of-freedom system whose equations of motion can be written in the compact form

\[ M \ddot{X} + C \dot{X} + K X = F. \]  

(1)

In the above relation, the symbol \( X \) denotes the time-dependent vector whose components are the displacements of the \( N \) degrees-of-freedom. Further, \( M, C \) and \( K \) represent the corresponding \( N \times N \) mass, damping and stiffness matrices. Finally, the \( N \)-component vector \( F \) denotes the external forces acting on the disk.

When the system is tuned, that is, when all the blades are perfectly identical, the matrices \( M, C \) and \( K \) possess a special structure which reflects the invariance of the disk under a rotation by an angle \( \frac{2\pi}{N} \). Interestingly, this property can be utilized to derive simple, closed form expressions for the natural frequencies, the mode shapes, and the steady state response of the disk to the harmonic excitation (Thomas, 1979; Mignolet and Christensen, 1990).

\[ F(t) = F_0^{(c)} \cos \omega t + F_0^{(s)} \sin \omega t. \]  

(2)

The introduction of mistuning in the system, for example in the form of small blade-to-blade variations in the mass distribution and natural frequencies, destroys the rotational symmetry of the disk and prevents the use of the corresponding closed form results in the determination of its dynamic response.

This analysis must then be accomplished by using standard techniques. In particular, the steady state response of the disk to the excitation vector given by Eq. (2) can be sought in the form

\[ X(t) = U \cos \omega t + V \sin \omega t. \]  

(3)

Then, it is readily shown that the components \( U_1 \) and \( V_1 \) of the constant vectors \( U \) and \( V \) satisfy the linear equations

\[ HZ = \dot{F}, \]  

(4)

with

\[ Z^T = [U_1 V_1 U_2 V_2 \ldots U_N V_N]. \]  

(5)

and

\[ \dot{F}^T = [F_0^{(c)} F_0^{(c)} F_0^{(s)} F_0^{(s)} \ldots F_0^{(c)} F_0^{(s)}] \]  

(6)

where \( ^T \) designates the operation of matrix transposition and \( H \) is a \( 2N \times 2N \) matrix whose block elements are composed of the terms \( \omega^2 M_{ij} \) and \( \omega C_{ij}, i,j=1,\ldots,N. \)

The lack of precise knowledge of the values of the mass and of the natural frequencies of the blades has led to the modeling of these quantities as random variables whose means correspond to the design specifications and whose small variances are representative of the manufacturing process. The presence of nondeterministic blade characteristics, \( M_0, C_0 \) and/or \( K_0 \) in Eq. (1) and (4) implies in turn that the components of the corresponding vectors \( U \) and \( V \) are also random variables. Their probabilistic properties must be determined for an accurate assessment of the vibration response of the mistuned bladed disk, in particular of the fatigue life.

RANDOM RESPONSE: COMBINED CLOSED FORM - PERTURBATION APPROACH

Probability Density Function of Amplitude : A Closed Form Expression

In this section, a closed form expression will be derived for the probability density function of the response of an arbitrary degree-of-freedom of a randomly mistuned bladed disk subjected to the harmonic excitation given by Eq. (2). For clarity of the presentation, assume that a total of \( m \) elements of the matrices \( M, C \) and \( K \) vary from a set of \( n \) blades to another one to such extent that they should be considered as random variables. Further, denote these parameters and their specified joint probability density function by \( \Lambda_i, i=1,\ldots,m \) and \( p_{\Lambda}(\lambda) = p_{\Lambda_1\Lambda_2\ldots\Lambda_m} (\lambda_1,\lambda_2,\ldots,\lambda_m) \), respectively.

Clearly, given a realization of the random blade parameters \( \Lambda_i \), the matrices \( M, C \) and \( K \) can be determined and the dynamic response of the disk, in particular the elements \( U_i \) and \( V_i \), can be computed from Eq. (4)-(6). Then, a sample realization of the amplitude \( \Lambda_i \) corresponding to the specified values of \( \Lambda \) is readily obtained as

\[ A_i = \sqrt{U_i^2 + V_i^2}. \]  

(7)

Viewed in this manner, the system of equations (4)-(7) appears as a transformation of the blade characteristics \( \Lambda_i \) into the amplitude of vibration \( \Lambda_i \).

Thus, following the rules of changes of random variables, a closed form expression can be derived for the probability density function of the amplitude \( p_{\Lambda}(a) \). Specifically, introducing the random variables \( \Lambda_j, j=1,\ldots,m \) defined as

\[ \Lambda_j = \Lambda_j, \quad j = 2,\ldots,m, \]  

(8a)

and

\[ \Lambda_j = \Lambda_j, \quad j = 2,\ldots,m, \]  

(8b)

it is readily shown that their joint probability density function \( p_{\Lambda}(\lambda) \) is

\[ p_{\Lambda}(\lambda) = \frac{p_{\Lambda}(\lambda)}{J(\Lambda_i \mid \Lambda)} \]  

(9)

where \( J(\Lambda_i \mid \Lambda) \) denotes the Jacobian of the transformation \( \Lambda_i \to \Lambda_i \).

It is straightforward to prove that this function reduces to

\[ J(\Lambda_i \mid \Lambda) = \left| \frac{\partial \Lambda_i}{\partial \Lambda_j} \right| = \frac{1}{2a} \left| \frac{\partial a}{\partial \Lambda_j} \right| \]  

(10)

where \( a \) designates the realized value of \( A_i \) corresponding to the selections \( \Lambda_j = \Lambda_j, i=1,\ldots,m \).

Finally, the probability density function \( p_{\Lambda}(a) \) is obtained by integrating the joint density \( p_{\Lambda}(\lambda) \) over the domain of the variables \( \Lambda_j = \Lambda_j, i=2,\ldots,m \), that is

\[ p_{\Lambda}(a) = 2a \int \cdots \int \frac{p_{\Lambda}(\lambda)}{J(\Lambda_i \mid \Lambda)} d\lambda_2 \cdots d\lambda_m. \]  

(11)

To complete the closed form determination of \( p_{\Lambda}(a) \), it
remains to derive an expression for the derivative \( \frac{\partial(a^2)}{\partial \lambda_i} \). To this end, denote by \( e_i \) the 2N-component vector whose elements are all zero except the \( i \)-th one which equals one. Then, it is found from Eq. (4) and (5) that
\[
U_i = e_i^T \xi_{Z-1} \quad Z = e_i^T \xi_{Z-1} \quad H^{-1} \quad \tilde{F}
\]
and
\[
V_i = e_i^T \xi_{Z} = e_i^T \xi_{Z-1} \quad H^{-1} \quad \tilde{F}
\]
so that
\[
a^2 = U_i^2 + V_i^2 = e_i^T \xi_{Z-1} \quad H^{-1} \quad \tilde{F} \quad \tilde{F}^T \quad H^{-1} \quad e_i^T \xi_{Z-1} + e_i^T \xi_{Z} \quad H^{-1} \quad \tilde{F} \quad \tilde{F}^T \quad H^{-1} \quad e_i = \xi_i \xi_i^T \quad H^{-1} \quad \tilde{F} \quad \tilde{F}^T \quad H^{-1} \quad e_i.
\]
In order to compute \( \frac{\partial(a^2)}{\partial \lambda_i} \), note that the derivatives with respect to a parameter \( \alpha \) of an arbitrary matrix \( D \) and of its inverse \( D^{-1} \) are related through the equation (Noble and Daniel, 1988)
\[
d(D^{-1}) = -D^{-1} \quad \frac{dD}{d\alpha} \quad D^{-1}.
\]
Thus, combining Eq. (14) and (15), it is found that
\[
\frac{\partial(a^2)}{\partial \lambda_i} = 2 \xi_i \xi_i^T \quad H^{-1} \quad \frac{\partial H}{\partial \lambda_i} \quad H^{-1} \quad \tilde{F} \quad \tilde{F}^T \quad H^{-1} \quad e_i + e_i^T \xi_i H^{-1} \quad \tilde{F} \quad \tilde{F}^T \quad H^{-1} \quad e_i\]
\[
+ e_i^T \xi_i H^{-1} \quad \frac{\partial H}{\partial \lambda_i} \quad H^{-1} \quad \tilde{F} \quad \tilde{F}^T \quad H^{-1} \quad e_i = \xi_i \xi_i^T \quad H^{-1} \quad \tilde{F} \quad \tilde{F}^T \quad H^{-1} \quad e_i\]
where the derivative \( \frac{\partial H}{\partial \lambda_i} \) is readily obtained by first determining which of the matrices \( H \), \( C \) and \( K \) involve \( \lambda_i \) and then locating these block elements in \( H \) according to Eq. (4)-(6).

Note finally that the procedure described in this section could also be employed to derive the joint probability density function of two, three or more amplitudes.

Probability Density Function of Amplitude : Perturbation Solution

From a purely mathematical point of view, Eq. (11), (14) and (16) form a complete and exact representation of the probability density function of the amplitude \( \lambda_i \) for any number of blades \( n \) and random characteristics \( \Lambda_i \). Indeed, Eq. (14) can be seen either as an expression for the squared amplitude \( a^2 \) in terms of \( \lambda_i, \ldots, \lambda_m \), or as an implicit definition of the variable \( \lambda_i \) as a function of \( a^2, \lambda_2, \ldots, \lambda_m \). This result can then be combined with Eq. (16) to provide an expression for \( \frac{\partial(a^2)}{\partial \lambda_i} \) involving only \( a^2, \lambda_2, \ldots, \lambda_m \), which in turn can be introduced in the denominator of the integrand of Eq. (11). Finally, upon evaluation of the \( m \)-fold integral, the probability density function \( p_{\Lambda_i}(a) \) is evaluated in closed form.

It should be recognized, however, that a direct application of the step by step procedure enunciated above involves simple mathematical operations which are computationally difficult to perform. The determination of the functional form of \( \lambda_i \) in terms of \( a^2, \lambda_2, \ldots, \lambda_m \) and the evaluation of the inverse \( H^{-1} \) are typical of this situation. In this context, note that quality control inspections lead to the rejection of blades whose characteristics are too different from the design specifications so that the standard deviation of the random variable \( \lambda_i, \sigma_i \), is small in comparison to its mean value, \( \bar{\lambda}_i \). Thus, the variables of integration \( \hat{\lambda}_i \) can be written in the form
\[
\hat{\lambda}_i = \bar{\lambda}_i + \sigma_i \quad \delta \lambda_i \quad i = 1, \ldots, m
\]
where \( \sigma_i \delta \lambda_i \) represents a small mistuning of the blade property \( \Lambda_i \) satisfying
\[
\sigma_i < \bar{\lambda}_i \quad i = 1, \ldots, m.
\]

The above inequality suggests a perturbation approach to the evaluation of the probability density function \( p_{\Lambda_i}(a) \). Specifically, decomposing the matrix \( H \) into a mean component \( \bar{H} \) corresponding to the design values \( \Lambda_i \) and a mistuned term \( \delta H \), it is found that
\[
H^{-1} = \left( \bar{H} + \delta H \right)^{-1} = \bar{H}^{-1} \left[ I + \bar{H} \delta H \bar{H}^{-1} \right]^{-1} = \bar{H}^{-1} \left[ I - \bar{H} \delta H \bar{H}^{-1} + (\bar{H} \delta H \bar{H}^{-1})^2 + \cdot \cdot \cdot \right]
\]
where \( \bar{I} \) denotes the \( 2N \times 2N \) unit matrix. Further, under the assumption that the design specifications correspond to a tuned bladed disk, the matrix \( \bar{H} \) possess a special structure which can be exploited to derive efficient techniques for the evaluation of \( \bar{H}^{-1} \) and of \( \delta H \) through Eq. (19) (see Sinha and Chen, 1988 for an example).

Note that the representation of \( H^{-1} \) given by Eq. (19) is valid only when the series \( I - \bar{H} \delta H \bar{H}^{-1} + (\bar{H} \delta H \bar{H}^{-1})^2 + \cdot \cdot \cdot \) converges, that is, provided that all of the eigenvalues of \( \bar{H} \delta H \bar{H}^{-1} \) belong to the open interval \((-1,1)\). Other series approximation of \( H^{-1} \) must be used if the domain of integration in Eq. (11) contains values of the deterministic parameters \( \lambda_1, \lambda_2, \ldots, \lambda_m \) to which corresponds a matrix \( \delta H \bar{H}^{-1} \) possessing at least one eigenvalue larger than one in modulus. In particular, note that the expansion
\[
H^{-1} = \left( \bar{H} + \delta H \right)^{-1} = \delta H^{-1} \left[ I - \bar{H} \delta H \bar{H}^{-1} + (\bar{H} \delta H \bar{H}^{-1})^2 + \cdot \cdot \cdot \right]
\]
is valid when all of the eigenvalues of \( \delta H \bar{H}^{-1} \) are outside the domain \((-1,1)\).

Introducing an appropriate limited Taylor expansion, such as Eq. (19) or (20), in Eq. (14) and (16), approximate expressions of \( \lambda_i \) and \( \frac{\partial(a^2)}{\partial \lambda_i} \) can be derived in terms of \( a^2, \lambda_i, \ldots, \lambda_m \) and a perturbation-like representation of \( p_{\Lambda_i}(a) \) can be obtained from Eq. (11).

The series expansion shown in Eq. (19) has already been suggested for the determination of the dynamic response of a mistuned bladed disk. Specifically, Sinha (1986) and Sinha and Chen (1988, 1989) have used this representation in connection with the random matrix \( H \) to derive the series solution of Eq. (4)
\[
Z = \bar{H}^{-1} \quad \tilde{F} \quad - \bar{H} \quad \delta H \quad \bar{H}^{-1} \quad \tilde{F} \quad + \cdot \cdot \cdot
\]
In the above equation, the tuned matrix \( \bar{H} \) corresponds as before to the design specifications \( \Lambda \) while \( \delta H \) involves the random variations of blade characteristics \( \Lambda - \Lambda \) as opposed to the deterministic differences \( \Lambda - \Lambda \) considered in the present approach. The convergence of the series solution (21) requires that the random terms \( \Lambda - \Lambda \) be "small" enough so that the eigenvalues of \( \delta H \bar{H}^{-1} \) all belong to the open interval \((-1,1)\). The term "small" should be used carefully in connection with random variables since such quantities can take on arbitrarily large values even for arbitrarily small variances provided that their probability density function extends to infinity.

Thus, the validity of Eq. (21) is limited to random blade characteristics \( \Lambda \) whose probability density is defined over a finite domain such that the eigenvalues of \( \delta H \bar{H}^{-1} \) all belong to the interval \((-1,1)\). This condition can be quite restrictive when the lowest eigenvalue of \( H \) is very small as occur for example when the frequency of the excitation is close to a natural frequency of the tuned bladed disk.

**EXAMPLE**

**Model**

The modalities of application of the combined closed form - perturbation approach will now be demonstrated by considering the simple, yet realistic, model of a bladed disk which is shown in...
considered in particular by Griffin and Sinha (1985), Sinha (1986), and blade-disk stiffnesses due to both the aerodynamic effects and the flexibility of the disk. Further, it is assumed that the masses and damping coefficients of the blades are all identical so that the mistuning is restricted to the blade-disk stiffnesses $k_i$, $i = 1, ..., n$. Thus, it is found that
\[ m = n = N \] (22)
and
\[ \Lambda_i = k_i + k_i, \quad i = 1, ..., n \] (23)
where $k_i$ designates the mean blade-disk stiffness common to all blades.

**Fig. 1 Model of Bladed Disk**

**Closed Form Expression**

To exemplify the derivation of the different terms present in Eq. (11), consider the evaluation of the probability density function of the first blade $p_i(a)$. The identification of the matrix $H$ is achieved first by rewriting the equations of motion of the bladed disk, Fig. 1, in the form of Eq. (4). It is readily found that (Lin, 1991)
\[
H = \begin{bmatrix}
    h_1 + k_1 & h_2 & h_3 & h_4 & 0 & 0 & \cdots & -h_2 & h_3 \\
    -h_2 & h_1 + k_1 & -h_2 & h_2 & 0 & 0 & \cdots & -h_2 & h_2 \\
    -h_3 & -h_2 & h_1 + k_2 & h_3 & -h_2 & h_4 & \cdots & 0 & 0 \\
    -h_4 & -h_2 & h_4 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
    -h_4 & -h_2 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\] (24)

where the deterministic parameters $h_1, h_2, h_3, h_4, k_1, k_2, k_3$ and $k_4$ are defined as
\begin{align}
    h_1 &= k_1 - m \omega^2 + 2 k_c \quad \text{(25)} \\
    h_2 &= k_c - n > 2 \quad \text{(26a)} \\
    h_3 &= 2 k_c - n = 2 \quad \text{(26b)} \\
    h_4 &= -(c + c_c) \omega \quad \text{(27)} \\
    k_1 &= c_c \omega \quad n > 2 \quad \text{(28a)} \\
    k_2 &= k_4 \omega \quad n = 2 \quad \text{(28b)}
\end{align}
The amplitude $a^2$ of response of the blade 1 corresponding to a set of stiffnesses $k_i$ is seen from Eq. (14) to be
\[
a^2 = \varepsilon_1^T H^{-1} \varepsilon_1 + \varepsilon_2^T H^{-1} \varepsilon_2 \] (29)
The evaluation of the Jacobian \[ \frac{\partial H}{\partial k_1} \] requires first the computation of the derivative \[ \frac{\partial H}{\partial k_1} \] which is readily achieved from Eq. (24) in the form
\[
\partial H = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} = \varepsilon_1 \varepsilon_1^T + \varepsilon_2 \varepsilon_2^T \] (30)

Then, introducing the above result in Eq. (16) and noting that (see Appendix I for a proof)
\[
\varepsilon_1^T H^{-1} \varepsilon_1 = \varepsilon_1^T H^{-1} \varepsilon_1 + \varepsilon_2^T H^{-1} \varepsilon_2 \]
and
\[
\varepsilon_2^T H^{-1} \varepsilon_2 = -\varepsilon_2^T H^{-1} \varepsilon_1 \]
it can be shown that the Jacobian reduces to
\[
\left| \frac{\partial (a^2)}{\partial k_1} \right| = 2 a^2 \left[ \varepsilon_1^T H^{-1} \varepsilon_1 \right]^2 \] (33)
In order to isolate the dependency of \[ \frac{\partial (a^2)}{\partial k_1} \] on the stiffness $k_1$, introduce the following partitions
\[
H = \begin{bmatrix}
    H_{11} & H_{12} \\
    H_{21} & H_{22}
\end{bmatrix}
\] (34)
and
\[
H^{-1} = \begin{bmatrix}
    L_{11} & L_{12} \\
    L_{21} & L_{22}
\end{bmatrix}
\] (35)
where the submatrices $H_{11}, H_{12}, H_{21}$ and $H_{22}$, or $L_{11}, L_{12}, L_{21}$ and $L_{22}$, have dimensions $2 \times 2$, $2 \times (2n-2)$, $(2n-2) \times 2$ and $(2n-2) \times (2n-2)$, respectively. Then, using the Frobenius-Schur formula for the inverse of a partitioned matrix, it is found that
\[
L_{11} = G^{-1} \] (36)
and
\[
L_{12} = -G^{-1} H_{12} H_{22}^{-1} H_{21} = -L_{11} H_{12} H_{22}^{-1} \] (37)
where
\[
G = H_{11} - H_{12} H_{22}^{-1} H_{21} = \begin{bmatrix}
    g_{11} + k_1 & g_{12} \\
    -g_{12} & g_{11} + k_1
\end{bmatrix} \] (38)
In the above equation, the symbols $g_{11}$ and $g_{12}$ designate two functions of the stiffnesses $k_2, ..., k_n$ defined by the relations
\[
g_{11} = h_1 - \varepsilon_1^T H_{12} H_{22}^{-1} H_{21} \varepsilon_1 \] (39)
and
\[
g_{12} = h_3 - \varepsilon_2^T H_{12} H_{22}^{-1} H_{21} \varepsilon_2 \] (40)
Finally, introducing the partitioned vector $W$ in the form
\[
W = \begin{bmatrix}
    I_2 & -H_{12} H_{22}^{-1} \\
    \varepsilon & \varepsilon_1 \varepsilon_1^T + \varepsilon_2 \varepsilon_2^T
\end{bmatrix}
\] (41)
it is found that (Lin, 1991)
\[
a^2 = \frac{W^2}{(g_{11} + k_1)^2 + g_{12}^2} \] (42)
and
\[
\frac{\partial (a^2)}{\partial k_1} = 2 a^2 \left[ \frac{W^2}{a^2} - \frac{W^2}{g_{12}^2} \right] \] (44)
where $W^2$ denotes the squared norm of the vector $W$, that is $W^T W$. Equations (11) and (42)-(44) form a complete and exact
representation of the probability density of the amplitude $A_1$. Note finally that this function can also be written in the form

$$P_{A_1}(A_1) = \frac{1}{2A_1^3 \tau} \left[ \frac{w^2}{A_1^2 g_{12}} \right]^{\frac{3}{2}}$$

(45)

where the expectation, $E\{\cdot\}$, is taken with respect to the variables $k_j, j=2,...,N$ under the constraint that $-\tau \leq k_1 \leq \tau$ or equivalently that

$$-\tau \leq g_{11} = \left[ \frac{w^2}{A_1^2 g_{12}} \right]^{\frac{3}{2}} \leq \tau$$

(46)

To complete the formulation of the problem, it remains to specify the joint probability density function of the stiffnesses $k_j$. For the sake of illustration, it has been assumed that these random variables are independent and identically, uniformly distributed so that

$$P_{k_1,k_2,...,k_n}(k_1,k_2,...,k_n) = \frac{1}{(2\pi)^\frac{n}{2}}$$

(47)

and 0 otherwise. Note that any other bonafide probability density function could have been selected since the combined closed form - perturbation method is valid for all probabilistic models of the blade's properties.

It can be shown that the rotational symmetry of both the system and the excitation implies that

$$P_{A_1}(A_1) = P_{A_2}(A_1) = \cdots = P_{A_n}(A_1)$$

(48)

so that Eq. (11) and (42)-(46) can in fact be used in connection with any blade.

Finally, to demonstrate the derivation of an approximate expression for the probability density function of the amplitude in both resonant and off-resonant conditions, the components of the force vector $F(t)$ are chosen in the form

$$F_i(t) = F_0 \cos(\omega t + \Psi_i)$$

(49)

where

$$\Psi_i = \frac{2\pi r}{n} (i-1).$$

(50)

When the parameter $\omega$ is selected as the $r^{th}$ natural frequency of the system given by

$$\omega_r = \sqrt{\frac{k_r + 4k_c \sin^2(\frac{\pi r}{n})}{m}}$$

(51)

the excitation $F_i(t)$ corresponds to the $r^{th}$ engine order and is proportional to the $r^{th}$ mode shape of the system. Off-resonant conditions can also be simulated from Eq. (49)-(50) by letting $\omega$ be different from the values $\omega_r$, Eq. (51).

Near-Resonant Excitation

The effects of mistuning are known to be especially important when the system is excited at or near one of its natural frequencies. Since the derivation of a reliable approximation of the probability density function of the response is simpler in the former case ($\omega = \omega_0$) than in the latter ($\omega = \omega_r$), a near-resonant excitation will first be considered. Further, the case of a small variance of the mistuned stiffnesses, or equivalently $\tau/k_1=1$, is of particular importance in this study since it corresponds to bladed disks manufactured under a tight quality control. Then, following a previous discussion, an approximate expression for the probability density function $P_{A_1}(A_1)$ will be sought by expanding the matrix $H_{22}$ according to Eq. (19) where $H_{22}$ is given by Eq. (24) and (34) with $k_2=k_3=\cdots=k_n=0$ and

$$\delta H_{22} = diag \left[ k_2, k_2, k_3, k_3, \cdots, k_n, k_n \right].$$

(52)

Keeping only the terms involving zeroth and first powers in $\delta H_{22}$, it is found that

$$g_{11} = (h_{11} - e_1^T H_{12} H_{22} H_{21} e_1) + (e_1^T H_{12} H_{22} H_{21} e_1) + O(\delta H_{22})$$

(53)

$$g_{12} = (h_{12} - e_1^T H_{12} H_{22} H_{21} e_2) + (e_1^T H_{12} H_{22} H_{21} e_2) + O(\delta H_{22})$$

(54)

where the scalars $g_{11}, g_{12}$ and vectors $g_{11}', g_{12}', g_{12}''$ are given in Appendix II. Proceeding in a similar manner in connection with the term $W^2$ leads to the expression

$$W^2 = w_0^2 + w_1^2 \delta H_{22} w_2$$

(55)

where the scalar $w_0$ and the vectors $w_1, w_2$ are given in Appendix II.

Finally, using Eq. (45), (53)-(55), a first order approximation of the probability density function can be expressed in the form

$$P_{A_1}(A_1) = \frac{w_0}{2\pi A_1^2 \sqrt{w_0 - A_1^2 g_{12}}} E \left[ \frac{w_0 - A_1^2 g_{12}}{2A_1^2} \right]^{\frac{1}{2}}$$

(56)

where the expectation $E\{\cdot\}$ is taken with respect to the variables $k_2, k_3, \cdots, k_n$ under the linearized version of the constraint (46), that is

$$-\tau \leq -g_{11} \pm \sqrt{\frac{w_0 - A_1^2 g_{12}}{2A_1^2}} \leq \tau.\ (57)$$

According to Eq. (56) and (57), the first order approximation of the probability density function $P_{A_1}(A_1)$ can be expressed as the expected value of a linear combination of the random variables $k_2, k_3, \cdots, k_n$ under the linear inequality constraints given by Eq. (57).

Thus, the value of $P_{A_1}(A_1)$ can readily be evaluated by relying on the algorithm recently developed by Mignolet and Lin (1991). For completeness, a summary of this computational technique is presented in Appendix III.
Resonant Excitation

The evaluation of the probability density function \( p_A(a) \) according to Eq. (11) or (45) is rendered more delicate in the case of a resonant excitation by the possible presence in the domain of integration, or in its neighborhood, of a singularity of the integrand. This situation is depicted in Fig. 2 in the simple, illustrative example of a 3-blade disk (the corresponding figure in the case of a 24-blade disk would be 23-dimensional). The surfaces denoted by CS and BS will be termed the critical and bound surfaces, respectively. They correspond to the conditions

\[
\frac{W^2}{a^2} - g_{12} = 0 \quad \text{for CS} \tag{58}
\]

and

\[
k_1 = g_{11} \pm \left[ \frac{W^2}{a^2} - g_{12} \right]^{1/2} \quad \text{for BS}. \tag{59}
\]

The region labeled 1 in Fig. 2 and which is comprised between the bound and critical surfaces is, for some values of \( r \) and \( n \), part of the domain of integration \( D \) corresponding to Eq. (46). This fact can be ascertained by computing the value of the stiffness \( k_1 \), Eq. (43), corresponding to any point \( P \) located between the bound and critical surfaces. Then, the region 1 is an integral part of the domain \( D \) if and only if the computed value of \( k_1 \) lies between \(-\tau\) and \( \tau \).

When necessary, the integration over the singular region 1 is accomplished in three steps (see Appendix IV for details). First, the functions \( W^2 \) and \( \frac{W^2}{a^2} - g_{12} \) are expanded in Taylor series of \( k_2, k_3, \ldots, k_n \) keeping respectively two and three terms (linear and quadratic approximations). Then, the function \( \frac{W^2}{a^2} - g_{12} \) is integrated in the direction perpendicular to the critical surface. This operation effectively removes the singularity of the problem. The third and final step of the computation which consists in the integration over the \( n - 2 \) tangential coordinates, is accomplished by relying, as in the off-resonant case on the algorithm described in Appendix III.

Although the integrand is well behaved in the remaining of the domain, i.e., the regions labeled 2-5, it should be noted that the singularity on the critical surface has a global effect on the function \( \frac{W^2}{a^2} - g_{12} \), producing in particular rapid, nonlinear changes in the normal direction to the critical surface. This behavior motivated the division of the domain of integration into a number of smaller regions, labeled 2-5 in Fig. 2. Further, the contribution of each of these regions to the probability density function \( p_A(a) \) has been obtained by relying first on a local linearization of the integrand with respect to the corresponding midpoint \( g_j \), see Fig. 2, and then on the algorithm described in the Appendix III.

**NUMERICAL EXAMPLE**

To demonstrate the effectiveness of the proposed combined closed-form-perturbation (CFP) method, a bladed disk has been considered whose parameters are as follows (Sinha and Chen, 1988)

\[
m = 0.0114 \quad K \tag{60}
\]

\[
k = 45430 \quad N/m \tag{61}
\]

\[
k = 430000 \quad N/m \tag{62}
\]

\[
c = 0 \tag{63}
\]

\[
c = 1.443 \quad N \cdot sec/m \tag{64}
\]

Further, it has been assumed that the mistuning originates in the stiffnesses, which are modeled as random variables uniformly distributed in the interval \( k_j \in [-8000N/m, 8000N/m] \). Shown in Fig. 3 and 4 are the probability density functions obtained by a thorough Monte Carlo simulation (curves labeled MC, 100,000 samples generated) and by Eq. (56) (curves labeled 56) corresponding to a 24-blade disk subjected to periodic excitations of frequencies \( \omega = 6000 \) and \( 6100 \), respectively. These excitation frequencies are close to the value \( \omega_p \), Eq. (51), which is readily found to be \( 6141.60 \) rad/sec. Clearly, the matching between the exact (Monte Carlo) and approximate (CFP) results is excellent in both cases. Note the presence on Fig. 4 of a small undulation in the CFP curve in the large amplitude tail of the distribution. This undulation is, from a numerical point of view, symptomatic of the proximity of the critical surface, Eq. (65), to the domain of integration and justifies the need for a separate investigation of the resonant conditions. Physically, the fluctuation indicates that some of the blades are undergoing resonance, and thus, display large amplitudes of response.

![Fig. 3 Probability Density of Blade Amplitude, \( \omega = 6000 \) rad/sec](image)

![Fig. 4 Probability Density of Blade Amplitude, \( \omega = 6100 \) rad/sec](image)
Also shown on Fig. 3 and 4 are the probability density functions obtained by relying on a three-term approximation of the equations of motion, as discussed by Sinha and Chen (1988, 1989). A comparison of these results clearly indicates that the combined closed form - perturbation approach provides a closer fit of the exact probability density function of the response.

Fig. 5 Probability Density, 16 blades, 3rd engine order

Fig. 6 Probability Density, 18 blades, 3rd engine order

Shown in Fig. 5-8 are the probability density functions, exact (through Monte Carlo simulation) and approximate (through the present approach), corresponding to 16, 18, 20 and 24-blade disks. It is seen that reliable approximations of $P_A(a)$ can also be obtained in the resonant case in spite of its aforementioned difficulties and peculiarities. It should be noted that the accuracy of the approximation obtained by the CFP method varies with the engine order. In particular, it appears that the zeroth engine order, which corresponds to excitation forces on the different blades which are in phase with each other, leads to the poorest matching between simulation and CFP results. This situation is depicted in Fig. 9 in context with a 24-blade disk. Finally, note that the ordinates on Fig. 5-9 have been scaled to satisfy the requirement that the total probability be equal to one.

Fig. 7 Probability Density, 20 blades, 3rd engine order

Fig. 8 Probability Density, 24 blades, 3rd engine order

Fig. 9 Probability Density, 24 blades, 0th engine order
Also shown on Fig. 5-9 are the approximate probability density functions obtained by relying on Sinha and Chen’s approach. A comparison of these curves with the corresponding results derived by the CFP method indicates that the combined closed form - perturbation technique performs always as well as and very often better than Sinha and Chen’s approach (Sinha and Chen, 1988, 1989) in modeling the probability distribution of the amplitude of response of mistuned bladed disks. This conclusion is reinforced by noting that the CFP results shown in Fig. 3-9 have been obtained by relying on linear approximations of $W^2$ and either linear or quadratic expansions of $a_2^2 - 8\Delta^2$ as opposed the cubic polynomial model of Sinha and Chen (1988, 1989).

CONCLUSIONS

In this paper the determination of approximate expressions for the probability density function of the amplitude of response of randomly mistuned bladed disk has been addressed. A new computational method, termed the combined closed form - perturbation approach has been suggested. First, it was recognized that the equations describing the steady state response of the disk, Eq. (4)-(6) represent a transformation between the set of random blade properties and the corresponding amplitudes of vibration. On the basis of this remark, an exact multi-fold integral representation of the probability density function of the blade amplitude was derived, Eq. (11), which is valid for any number of blades and any type of mistuning. Next, the computational aspects of the determination of the probability density function from this representation were investigated. In particular, it was found that the small magnitude of the variance of the blade properties could lead to simple perturbation-like approximations by relying on Eq. (19).

The modalities of application of the combined closed form - perturbation approach were demonstrated by considering a simple model of the disk in which each blade is represented by a single-degree-of-freedom system. Accurate approximations of the probability density function of the amplitude of response were sought first for near-resonant excitations. A reliable approximation of the probability density function was derived on the basis of a small mistuning, Eq. (56), through a series expansion of the integral probability density function was derived on the basis of a small mistuning, Eq. (11), which is valid for any number of blades and any type of mistuning. Next, the computational aspects of the determination of the probability density function from this representation were investigated. In particular, it was found that the small magnitude of the variance of the blade properties could lead to simple perturbation-like approximations by relying on Eq. (19).

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APPENDIX I

The special structure of the matrices $H$ and $\tilde{S}H$ provides a series of simplifications in the determination of the Jacobian, Eq. (33). Specifically, partitioning $(H+\tilde{S}H)$ as a $N\times N$ block matrix, it is readily seen that each of its $2\times 2$ block element, denoted here by

$$(H+\tilde{S}H)_{jl},$$

can be written in the form

$$(H+\tilde{S}H)_{jl} = \begin{bmatrix} d & e \\ -e & d \end{bmatrix}$$

(A.1)

for some values of the coefficients $d$ and $e$. Such $2\times 2$ matrices, termed here anticentrosymmetric, are characterized by the condition

$$P_2^{-1}(H+\tilde{S}H)_{jl} P_2 = (H+\tilde{S}H)_{jl}$$

(A.2)

where

$$P_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(A.3)

In fact, it is readily shown that Eq. (A.2) is a necessary and sufficient condition for the existence of the representation (A.1) of the matrix $(H+\tilde{S}H)_{jl}$. Since the property (A.2) is valid for all $2\times 2$ block elements of $(H+\tilde{S}H)$, it is readily shown that this matrix satisfies the equation

$$P_2^{-1}(H+\tilde{S}H)_{jl} P_2 = (H+\tilde{S}H)_{jl}$$

(A.4)

where

$$P_N = \begin{bmatrix} P_2 & 0 & \ldots & 0 \\ 0 & P_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ldots & \ldots & \ldots & P_2 \end{bmatrix}$$

(A.5)

It is now possible to show that the matrix $(H+\tilde{S}H)^{-1}$ also satisfies Eq. (A.4) and

$$(H+\tilde{S}H)(H+\tilde{S}H)^{-1} = I_{2N}$$

(A.6)

and

$$P_N^{-1} P_N = P_N$$

(A.7)

where $I_{2N}$ denotes the $2N\times 2N$ identity matrix. Then, rewrite Eq. (A.6) in the form

$$P_N^{-1}(H+\tilde{S}H)(P_N^{-1})(H+\tilde{S}H)^{-1} P_N = P_N^{-1} I_{2N}$$

or equivalently as

$$P_N^{-1}(H+\tilde{S}H) P_N = I_{2N}$$

(A.9)

In view of the property (A.4), it is found that

$$(H+\tilde{S}H) P_N (H+\tilde{S}H)^{-1} P_N = I_{2N}$$

(A.10)

so that

$$P_N^{-1}(H+\tilde{S}H)^{-1} P_N = (H+\tilde{S}H)^{-1}$$

(A.11)

The last equation shows that $(H+\tilde{S}H)^{-1}$ also satisfies Eq. (A.4) and thus, that each of its $2\times 2$ block elements $$(H+\tilde{S}H)^{-1}_{jl}$$

is anticentrosymmetric. Equations (31) and (32) then follow by considering in particular $$(H+\tilde{S}H)^{-1}_{11}$$

APPENDIX II

It is readily seen from Eq. (53)-(55) that

$$g_{11}^0 = k_1 - m \omega^2 + 2 k_2 - \xi_1^T H_{12} H_{22}^{-1} H_{21} \xi_1$$

(A.12)

$$g_{11}^R = H_{22}^{-1} H_{21} \xi_1$$

(A.13)

$$g_{11}^L = \xi_1^T H_{12} H_{22}^{-1} H_{21} \xi_2$$

(A.14)

$$g_{11}^0 = -c_\omega \omega - \xi_1^T H_{12} H_{22}^{-1} H_{21} \xi_2$$

(A.15)

$$g_{12}^R = H_{22}^{-1} H_{21} \xi_2$$

(A.16)

Further, introducing the partition

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

(A.17)

where $F$ is a 2-component vector, it is found from Eq. (41) that

$$W = F_1 - H_{12} H_{22}^{-1} F_2 + O(\tilde{S} H_{22})$$

(A.18)

so that
\[
W^2 = W^T W = \left[ \tilde{E}_1 - H_{12} \tilde{H}_{22} \tilde{E}_2 \right]^T \left[ \tilde{E}_1 - H_{12} \tilde{H}_{22} \tilde{E}_2 \right] + 2 \left[ \tilde{E}_1 - H_{12} \tilde{H}_{22} \tilde{E}_2 \right]^T H_{12} \tilde{H}_{22} \tilde{E}_2 + O \left( \tilde{H}_{22}^2 \right). 
\]

(A.19)

The above expression can be written in the form of Eq. (55) by selecting
\[
w_0 = \left[ \tilde{E}_1 - H_{12} \tilde{H}_{22} \tilde{E}_2 \right]^T \left[ \tilde{E}_1 - H_{12} \tilde{H}_{22} \tilde{E}_2 \right] + 2 \left[ \tilde{E}_1 - H_{12} \tilde{H}_{22} \tilde{E}_2 \right]^T H_{12} \tilde{H}_{22} \tilde{E}_2 + O \left( \tilde{H}_{22}^2 \right). 
\]

(A.20)

and
\[
w_1 = 2 \left[ \tilde{E}_1 - H_{12} \tilde{H}_{22} \tilde{E}_2 \right]^T H_{12} \tilde{H}_{22} \tilde{E}_2 + O \left( \tilde{H}_{22}^2 \right). 
\]

(A.21)

APPENDIX III

The evaluation of the moments \( E \left[ k_i \right] \sum_{j=1}^{n} b_j k_j \leq b_0 \) of uniform random variables \( k_j \) satisfying a linear inequality constraint of the form \( \sum_{j=1}^{n} b_j k_j \leq b_0 \) has recently been investigated by Mignolet and Lin (1991). It was shown in particular that
\[
E \left[ \sum_{i=1}^{n} a_i k_i \right] = \sum_{i=1}^{n} a_i E \left[ k_i \right] \sum_{j=1}^{n} b_j k_j \leq b_0 
\]

\[
= \frac{1}{n+1} \sum_{i=1}^{n} b_i \left[ d_0 b_0 - \sum_{i=1}^{n} (d_0 - d_i) \left[ b_0 - b_j \right] \right] 
+ \sum_{i,j=1}^{n} \left( d_0 - d_i - d_j \right) \left[ b_0 - b_j - b_j \right] 
\]

(A.23)

where the notation \( \lfloor u \rfloor \) denotes the maximum of \( u \) and 0. Further, \( d_0 = \frac{b_0}{n+1} \sum_{i=1}^{n} a_j \) (A.24a)

and
\[
d_i = -a_i + \frac{b_i}{n+1} \sum_{j=1}^{n} a_j. 
\]

(A.24b)

A reduction of the number of computations required to evaluate
\[
E \left[ \sum_{i=1}^{n} a_i k_i \mid \sum_{j=1}^{n} b_j k_j \leq b_0 \right] 
\]

according to Eq. (A.23) can be achieved when some of the coefficients \( b_j \) are small. Then, separating the parameters \( b_j \) in two groups, the large coefficients \( b_{j}, j = 1, \ldots, n-m \) and the small ones
\[
c_j = b_{n-m+j} \quad j = 1, \ldots, m, 
\]

it is found that
\[
E \left[ \sum_{i=1}^{n} a_i k_i \mid \sum_{j=1}^{n} b_j k_j \leq b_0 \right] = \frac{1}{\prod_{i=1}^{n} b_i} \left( n-m \right)! \left[ e_0^{(1)} b_0^{n-m-1} \sum_{i=1}^{n} (e_0^{(1)} - e_i^{(1)}) \left[ b_0 - b_j \right] \right] 
+ \sum_{i,j=1}^{n} (e_0^{(1)} - e_i^{(1)}) \left[ b_0 - b_i - b_j \right] (A.24b)
\]

(A.25)

APPENDIX IV

Let \( K = [k_2, k_3, \ldots, k_p]^T \) be the position vector of an arbitrary point \( P \) in the region labeled 1 (see Fig. 2). Similarly, let \( K_Q \) denote the position vector of a point \( Q \) located on the critical surface. Then, expanding in Taylor series the function \( W^2 \) and \( W^2 \) with respect to the point \( K_Q \), it is found that
\[
W^2 = \mu_0 + W^2 \left( K - K_Q \right) + \left( K - K_Q \right)^T B_1 \left( K - K_Q \right) 
\]

(A.29)

\[
W^2 = \frac{a^2 - g_1^2}{a^2} \left( K - K_Q \right) + \left( K - K_Q \right)^T B_2 \left( K - K_Q \right) 
\]

(A.30)

where \( \mu_0 \) is the value of the function \( W^2 \) corresponding to \( K_Q \). Further, \( B_1 \) and \( B_2 \) are the gradients and Hessians of the functions \( W^2 \) and \( W^2 \) with respect to the point \( K_Q \). Note that the vector \( B_2 \) is in the direction normal to the surface \( \frac{a^2 - g_1^2}{a^2} = 0 \) at the point \( K_Q \). Further, define the matrix \( D \) as
\[
D = \begin{bmatrix}
u_2 & u_3 & u_4 & \ldots & u_n \\
u_3 & -u_2 & 0 & \ldots & 0 \\
u_4 & 0 & -u_2 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
u_n & 0 & 0 & \ldots & -u_2 
\end{bmatrix}
\]

(A.31)

and introduce the new position vector \( \alpha = [\alpha_2, \alpha_3, \ldots, \alpha_n]^T \)
defined by
\[ K = K_Q + D \alpha. \] (A.33)

Then, the contribution of the domain 1 to the probability density function can be expressed as a multifold integral over the variables \( \alpha_2, \alpha_3, \ldots, \alpha_n \). A Taylor expansion of the corresponding integrand with respect to the normal variable \( \alpha_2 \) yields the following approximate contribution

\[ \det(D) \int \left[ \frac{d_2 \alpha_2 + d_2}{\sqrt{d_3 \alpha_3^2 + d_4 \alpha_4 + d_5}} \right] d \alpha_3 \ldots d \alpha_n \] (A.34)

where \( \varepsilon \) is the small distance between the critical and the first bound surfaces. Further, the functions \( d_1, d_2, d_3, d_4 \) and \( d_5 \) are quadratic in the variables \( \alpha_i, i = 3, 4, \ldots, n \). The integration over \( \alpha_2 \) can be evaluated by standard techniques as (Gradshteyn and Ryzhik, 1990)

\[ (d_2 - \frac{d_4 d_1}{2 d_3}) \frac{1}{\sqrt{d_3}} \ln \left(2 \sqrt{d_3 R} + 2 d_3 \alpha_2 + d_4 \right) \text{ when } d_3 > 0 \] (A.35a)

and

\[ (d_2 - \frac{d_4 d_1}{2 d_3}) \frac{-1}{\sqrt{-d_3}} \sin^{-1} \left(\frac{2 d_3 \alpha_2 + d_4}{\sqrt{d_3^2 - 4 d_3 d_5}}\right) \text{ when } d_3 < 0 \] (A.35b)

where \( R = d_3 \alpha_3^2 + d_4 \alpha_4 + d_5 \). A final linearization of Eq. (A.35) with respect to the variables \( \alpha_3, \alpha_4, \ldots, \alpha_n \) yields the following approximation

\[ \int \frac{d_2 \alpha_2 + d_2}{\sqrt{d_3 \alpha_3^2 + d_4 \alpha_4 + d_5}} d \alpha_3 = E_0 + \sum_{i=3}^n E_i \alpha_i. \] (A.36)

It remains then to return to the original variables \( k_i \) and to compute the remaining integral. The first step is readily accomplished by noting that

\[ \alpha' = D'(K - K_Q) \] (A.37)

where

\[ \alpha' = [\alpha_3, \alpha_4, \ldots, \alpha_n]^T \] (A.38)

and

\[ D' = \begin{bmatrix} \text{2nd row of } D^{-1} \\ \text{3rd row of } D^{-1} \\ \vdots \\ \text{last row of } D^{-1} \end{bmatrix} \] (A.39)

While the evaluation of the integral over the variables \( k_3, \ldots, k_n \) is achieved by relying on the algorithm described in Appendix III.