An Unsteady Lifting Surface Theory for Ducted Fan Blades

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ABSTRACT

A frequency domain lifting surface theory is developed to predict the unsteady aerodynamic pressure loads on oscillating blades of a ducted subsonic fan. The steady baseline flow as observed in the rotating frame of reference is the helical flow dictated by the forward flight speed and the rotational speed of the fan. The unsteady perturbation flow, which is assumed to be potential, is determined by solving an integral equation that relates the unknown jump in perturbation velocity potential across the lifting surface to the upwash velocity distribution prescribed by the vibratory motion of the blade. Examples of unsteady pressure distributions are given to illustrate the differences between the three dimensional lifting surface analysis and the classical two dimensional strip analysis. The effects of blade axial bending, bowing (i.e., circumferential bending) and sweeping on the unsteady pressure load are also discussed.

NOMENCLATURE

\( a \) undisturbed speed of sound
\([A]\) influence matrix
\( b_n \) mode coefficient
\( C \) or \( C_r \) blade chord at constant radius \( r \)
\( C_a \) axial chord
\( d_n \) coefficient of shifted Chebyshev polynomial of order \( n \)
\( \bar{e}_\theta \) unit vector in \( \theta \) direction
\( \bar{e}_z \) unit vector in \( z \) direction
\( f(x) \) displacement normal to the helical chord at constant radius
\( g(\tilde{r}, \tilde{r}) \) cascade source solution
\( G(\tilde{r}, \tilde{r}) \) duct source solution
\( H_n \) potential mode function
\( i \) unit vector in \( x \) direction
\( K \) kernel function
\( K_s \) singular part of kernel function
\( LE \) leading edge
\( M \) axial flow Mach number
\( M_r \) relative flow Mach number defined by \([1 + (\Omega r/U)^2]^{1/2} \)
\( \hat{n} \) unit vector normal to helical chord pointing into fluid from suction side
\( N_b \) number of blades
\( N_d \) number of nodal diameters
\( N_m \) number of mode functions
\( p \) perturbation pressure
\( \bar{q} \) velocity vector
\( r \) radial coordinate
\( \bar{r} \) upwash point
\( r_0 \) load point
\( \text{real} \) real part
\( \text{R} \) compressible radius
\( R_{hi} \) fan hub radius, i.e. blade root radius
\( R_f \) fan tip radius, i.e. fan case radius
\( S_o \) surface area
\( T_n \) shifted Chebyshev polynomial of order \( n \)
\( \text{TE} \) trailing edge
\( U \) forward flight speed of fan rotor
\( U_r \) relative flow velocity defined by \([U^2 + (\Omega r)^2]^{1/2} \)
\( W \) upwash velocity
\( x \) local coordinate along helical chord measured from leading edge
\( y \) local coordinate normal to helical chord at constant radius
\( y' \) local coordinate normal to helix and tangent to helical surface
\( z \) engine axis coordinate, positive looking rearward
\( \alpha \) stagger angle; \( \tan \alpha = U/\Omega r \)
\( \beta_{rel} = (1 - M_{rel}^2)^{1/2} \)

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Δ jump across helical surface
θ circumferential coordinate whose positive direction is opposite to the direction of rotation
π 3.1415926
ρ undisturbed air density
σ(r) helix number as a function of r
φ perturbation velocity potential
ω oscillation frequency
Ω rotational speed of fan rotor
∇² Laplacian operator

Subscript
n mode index
r at radial location with radius r
rel relative
R real part
I imaginary part

INTRODUCTION

Advanced commercial and military engines have low aspect ratio fan blades that are often highly swept. Aeroelastic analysis of these fan blades using the classical two dimensional strip theory, which was developed for high aspect ratio straight blades, becomes fundamentally questionable. In fact, even for high aspect ratio straight blades, the two dimensional strip theory is inherently limited for those vibration problems that exhibit large spanwise gradient of vibration amplitude.

Analytical prediction of three dimensional unsteady aerodynamic flows associated with vibration of ducted rotating blades was pursued by many researchers. McCune (1958a, 1958b, 1972), and Okurummu and McCune (1970, 1974) studied steady and unsteady lifting line and lifting surface theories for subsonic and transonic flows with limited results. Homicz and Lordi (1981), Lordi and Homicz (1981) studied the steady loading problem for subsonic relative flows. Namba (1974, 1977), Namba and Ishikawa (1983), Kodama and Namba (1989) studied the subsonic and transonic unsteady flow cascade problems for helical blades. In all these work, the sonic cylinder at part span location is considered, but none allows the steady mean flow nonuniformity in the flow field for unsteady pressure load calculations.


In a transonic theory formulation for ducted fan blades, Chi (1985, 1986) developed an analytical framework that treats the nonuniform mean flow velocity field as a spatial distribution of simple harmonic sources and doublets. It is essentially a generalization of the transonic small disturbance theory for the fixed wing problem by Landahl (1961) to the rotating fan blade configuration. To maintain the analytical framework of the well established transonic small disturbance theory, the unsteady perturbation velocity potential is used by Chi (1985) as the explicit unknown variable. The unsteady perturbation pressure is simply related to the perturbation potential by the linearized Bernoulli's equation. This makes Chi's theory differ from others that directly choose the unsteady perturbation pressure as the unknown variable. To establish the solution technique for solving the perturbation velocity potential together with the perturbation pressure, the transonic axial flow problem that corresponds to a constant subsonic relative flow for each radial station was investigated in detail by Chi (1990a). The subsonic theory, solution procedure, and representative results are given in this paper.

THEORY

Helical Baseline Flow

As shown in Fig. 1, the helical flow pattern specified by the forward flight speed and the fan rotational speed is taken as the baseline steady flow about which all flow perturbations are defined. Throughout the paper, the fan case radius R is used to non-dimensionalized all coordinates, parameters and physical variables as required. This is equivalent to set R to unity.

At each radial location, a helix can be drawn with a pitch measured in the axial flow direction equal to the axial flow velocity multiplied by 2π and divided by the rotational speed Ω. A three dimensional helical blade can be generated by specifying first an arbitrary space curve called the helical blade generator inside the flow annulus originating from a point on the hub surface and extending to the fan case surface, and then assigning at each radial station the leading edge and trailing edge locations along its helix. The mathematical representation of the helical blade is given by

$$z = \frac{U}{\Omega} \theta + \sigma(r)$$  \hspace{1cm} (1)

where σ(r) is the helix number that can be interpreted as the axial shift of the helix at a radial station relative to the origin of the cylindrical coordinate system fixed to the spinning rotor (see Figure 2). The helical blade generator can be made to coincide with, for example, the leading edge locus, the midchord locus, or the trailing edge locus of a real fan blade.

The helical blade generated for a forward flight condition according to the procedure described above will not cause flow disturbances in an inviscid flow, because the mean flow velocity is everywhere tangent to the helical surface. Flow disturbances can only be produced by sources such as blade thickness, camber, incidence angle, blade vibration, flow path convergence, and flight vehicle pitch and yaw motion. The inclusion of these disturbance sources in formulating transonic flow analysis is dis-
cussed by Chi (1985). Here, we consider exclusively perturbations due to vibration of unloaded blades by leaving out other disturbance sources.

**Governing Equations**

For an observer in the cylindrical coordinate system fixed to the rotating fan, the partial differential equation that governs the perturbation velocity potential \( \phi \) for an oscillation frequency \( \omega \) is the convective wave equation:

\[
\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0
\]

(2)

where

\[
\nabla^2 = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}
\]

\( \mathfrak{P} = i \omega + \Omega \frac{\partial}{\partial \theta} + U \frac{\partial}{\partial z} \)

The unsteady perturbation pressure is related to the perturbation velocity potential by the linearized Bernoulli’s equation:

\[
p = -\rho \left( i \omega \phi + \Omega \frac{\partial \phi}{\partial \theta} + U \frac{\partial \phi}{\partial z} \right)
\]

(3)

where \( U \) is the relative flow velocity at a radial station, and \( \frac{\partial}{\partial x} \) refers to the partial derivative along the helical chord.

**Boundary Conditions**

The boundary conditions required to solve the governing equation (2) are the following:

(a) In the direction normal to the helical blade surface, the induced fluid particle velocity must match the blade vibration velocity. For an inviscid flow, the fluid particle is allowed to slide along the blade surface.

(b) The jump in perturbation pressure must vanish at the blade trailing edge (Kutta condition).

(c) The fluid velocity normal to the hub and casing surfaces must vanish.

(d) The unsteady flow disturbances must decay or propagate away from the fan in the rotating reference frame.

**Integral Equation**

By applying integral theorems, the partial differential equation (2) in conjunction with the boundary conditions, one can derive the following integral equation that relates the blade upwash velocity \( W(\tilde{r}) \) to the jump in perturbation velocity potential \( \Delta \phi \) across a reference blade:

\[
W(\tilde{r}) = \int \int K(\tilde{r}_0, \tilde{r}) \Delta \phi (\tilde{r}_0) \, d\tilde{s}_0
\]

(4)

where \( \Delta \phi \) is the jump in potential across a reference blade and its helical wake extending to downstream infinity. Here, the upwash velocity \( W \) is defined by

\[
\frac{W(x)}{U} = \frac{\sigma f_U [1 + (\Omega/U)^2]^{1/2} + i \gamma_U}{\left[ 1 + \frac{\sigma (f_U)^2}{(U)^2} \right]^{1/2}}
\]

where \( f(x) \) is the displacement normal to the helical chord and tangent to the constant radius cylinder at a given radial station, \( \frac{\partial f}{\partial x} \) is the chordwise slope of the displacement function, \( \frac{df}{dr} \) represents the rate of twist variation along the blade span. The kernel function \( K(\tilde{r}_0, \tilde{r}) \), which can be called the potential kernel, is related to the pressure kernel used by Namba (1974), and Namba and Ishikawa (1983) by a Fourier type integral along the helix from the load point to downstream infinity. See Chi (1990b) for the detailed proof.

**Kernel Function**

The “potential kernel” \( K(\tilde{r}_0, \tilde{r}) \) is defined by

\[
K(\tilde{r}_0, \tilde{r}) = \frac{\sigma g}{\partial n \partial n_0} (\tilde{r}_0, \tilde{r})
\]

(5)

where the cascade source solution \( g(\tilde{r}_0, \tilde{r}) \) for an interblade phase angle \( \sigma \) is given by

\[
g(\tilde{r}_0, \tilde{r}) = \sum_{k=1}^{N_k} \epsilon^{i k \theta} G(\tilde{r}_0, \theta_0 + [k - 1] \frac{2\pi}{N_k}, \tilde{r}, \theta, z)
\]

(6)

Here, the Green’s function \( G(\tilde{r}, \tilde{r}_0) \) with \( \tilde{r} \) and \( \tilde{r}_0 \) interchanged for the annular duct that satisfies the zero radial gradient condition on the hub and case surfaces is governed by the partial differential equation

\[
\nabla^2 G - \frac{1}{a^2} \frac{\partial^2 G}{\partial z^2} = \delta (\tilde{r} - \tilde{r}_0)
\]

(7)

where the adjoint operator \( \mathfrak{P} \) for the reversed flow is defined by

\[
\mathfrak{P} = i \omega - \Omega \frac{\partial}{\partial \theta} - U \frac{\partial}{\partial z}
\]

Equation (7) can be solved by performing a Fourier series expansion in \( \theta \) and a Fourier transform in \( z \) followed by a direct
solution of the resultant Bessel's equation in \( r \). The mathematical details are given by Chi (1986). The result is

\[
G (r, r_0, \theta - \theta_0, z - z_0) = \left( \frac{1}{2\pi} \right)^2 \sum_{n=\pm \infty}^\infty e^{i(n+\theta_0-z_0)} \int_{-\infty}^{\infty} dy_n (r, r_0, \alpha) e^{i\alpha z}.
\]

\[
y_n(r, r_0, \alpha) = \left\{ \begin{array}{ll} F_1(r) F_2(r_0) & \text{for } r < r_0 \\ F_1(r_0) F_2(r) & \text{for } r > r_0 \end{array} \right.
\]

\[
F_1 (r) = u_1 (r) u_2 (R_1) - u_2 (r) u_1 (R_1)
\]

\[
F_2 (r) = u_1 (r) u_2 (R_2) - u_2 (r) u_1 (R_2)
\]

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<thead>
<tr>
<th>( u_1' (r) )</th>
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\[
b_0 = \frac{1}{M^2 - 1} \left[ \alpha - \frac{M}{1 + M \left( \frac{\omega}{U} - \frac{\Omega}{U} \right)} \right] \left[ \alpha + \frac{M}{1 - M \left( \frac{\omega}{U} - \frac{\Omega}{U} \right)} \right]
\]

Here, the \( u_1' \) and \( u_2' \) represent the derivatives of \( u_1 \) and \( u_2 \) with respect to \( r \).

As shown in equation (6) the cascade source solution \( g(R_0, \vec{r}) \) is the Fourier sum of the duct source solution \( G(R_0, \vec{r}) \). Because of the complex Fourier series format for the \( \theta - \theta_0 \) part of the duct Green's function \( G(R, \vec{r}) \) and the blade summation \( (i.e., k \text{ summation}) \) part of the cascade Green's function \( g(R_0, \vec{r}) \), it is shown below that the summation of circumferential waves \( (i.e., \text{summation over } n) \) is limited to certain \( n \) values depending on two factors: the nodal diameter (ND) associated with the prescribed interblade phase angle \( \alpha \), and the number of blades in the fan rotor \( (N_b) \). Substituting equation (8) into equation (6) and noting the interchange of \( \vec{r} \) and \( \vec{r}_0 \) in the substitution, one obtains

\[
g (R_0, \vec{r}) = \left( \frac{1}{2\pi} \right)^2 \sum_{k=1}^{N_b} e^{i(k-1)\theta} e^{i\theta_0} \int_{-\infty}^{\infty} dy_n (R_0, r, \alpha) e^{i\alpha z (z_0 - z)}
\]

\[
= \left( \frac{1}{2\pi} \right)^2 \sum_{n=-\infty}^{\infty} \sum_{k=1}^{N_b} e^{i(k-1)(\alpha + \frac{\Omega}{U})} \int_{-\infty}^{\infty} dy_n (R_0, r, \alpha) e^{i\alpha z (z_0 - z)}
\]

The factor inside the square bracket must be equal to \( N_b \) or 0, i.e.

\[
\sum_{k=1}^{N_b} e^{i(k-1)(\alpha + \frac{\Omega}{U})} = \begin{cases} N_b & \text{if } n = -ND + m \cdot N_b \\ 0 & \text{otherwise} \end{cases}
\]

where the interblade phase angle \( \alpha \) is related to the nodal diameter ND by

\[
\alpha = 2\pi \frac{ND}{N_b}
\]

Therefore, the cascade Green's function becomes

\[
g (R_0, \vec{r}) = \left( \frac{1}{2\pi} \right)^2 \cdot \sum_{n=-\infty}^{\infty} \sum_{k=1}^{N_b} e^{i\theta_0} \int_{-\infty}^{\infty} dy_n (R_0, r, \alpha) e^{i\alpha z (z_0 - z)}
\]

which is basically \( N_b \) times the duct Green's function \( G(R_0, \vec{r}) \) except that the summation over the circumferential wave number \( n \) no longer covers all possible integer values from \( -\infty \) to \( \infty \) and instead covers only those integer \( n \) values that satisfy the following relationship

\[
n = -ND + m \cdot N_b
\]

where \( m = 0, \pm 1, \pm 2, \ldots \)

In equation (5) that defines the potential kernel function, the normal derivative is simply the scalar product between a gradient vector quantity and the unit normal vector. Therefore, the kernel
The three components $n_r$, $n_\theta$, and $n_z$ for a helical surface described by equation (1) are simply:

$$
n_r = -\frac{\partial \phi}{\partial r} \left| \frac{1}{\nabla F} \right|,
$$

$$
n_\theta = -\frac{U}{\Omega r} \left| \frac{1}{\nabla F} \right|,
$$

$$
n_z = \frac{1}{\left| \nabla F \right|}.
$$

Here the unit normal components $n_r$, $n_\theta$, and $n_z$ for a helical surface defined similarly.

Besides its dependence on the load position $\tilde{r}_0$ and the upwash location $\tilde{r}$, the potential kernel function is also a function of the axial flow Mach number $M$, the nondimensional rotor speed $\Omega/U$, the nondimensional oscillation frequency $\omega/U$, the number of blades $N_b$, and the number of nodal diameters $N_o$.

Furthermore, in the cylindrical coordinate system the dependence of the kernel function on $\tilde{r}_0$ and $\tilde{r}$ appears as $K(\tilde{r}_0, \tilde{r})$ as follows:

$$
K(\tilde{r}_0, \tilde{r}) = \begin{pmatrix}
\frac{\partial^2 g}{\partial r \partial r} & \frac{1}{r_0} \frac{\partial^2 g}{\partial \theta \partial r} & \frac{1}{r_0} \frac{\partial^2 g}{\partial r \partial \theta} \\
\frac{1}{r_0} \frac{\partial^2 g}{\partial \theta \partial r} & \frac{1}{r_0} \frac{\partial^2 g}{\partial \theta \partial \theta} & \frac{1}{r_0} \frac{\partial^2 g}{\partial \theta^2} \\
\frac{1}{r_0} \frac{\partial^2 g}{\partial r \partial \theta} & \frac{1}{r_0} \frac{\partial^2 g}{\partial \theta^2} & \frac{1}{r_0} \frac{\partial^2 g}{\partial \theta^2}
\end{pmatrix} \begin{pmatrix}
n_r \\
n_\theta \\
n_z
\end{pmatrix}
$$

The three components $n_r$, $n_\theta$, $n_z$ are defined similarly.

First of all, the reference blade and wake surfaces are discretized into finite size constant potential elements. The potential jump values for the wake elements are related to the potential jump value at the blade trailing edge for the same radius. This is because the pressure jump vanishes in the wake, and consequently the linearized Bernoulli's equation (3) can be integrated to relate the potential in the wake to that at the blade trailing edge, i.e.

$$
\Delta\phi(x) = \Delta\phi(x_{TE}) e^{i\omega t_{(n/\pi)}} \text{ for } x > x_{TE}
$$

where $x$ denotes the distance measured along the helix at a given radial station. Physically, this corresponds to vortex convection along the helical wake with the relative flow velocity $U_r$. Therefore, the only unknowns are the potential jumps for all the blade elements.

The jump in perturbation velocity potential $\Delta \phi$ for a given radial station, is then expressed as a finite series expansion in terms of selected potential mode functions $H_n(x/c_r)$:

$$
\frac{\Delta \phi(x)}{U_r c_r} = -\frac{N_o}{\pi} \sum_{n=0}^{N_o-1} b_n H_n \left( \frac{x}{c_r} \right)
$$

The corresponding jump in perturbation pressure is also represented as a finite series expansion in terms of selected pressure mode functions:

$$
\frac{\Delta p(x)}{\rho U_r^2} = \left( \frac{1-x/c_r}{x/c_r} \right)^{1/2} \sum_{n=0}^{N_o-1} b_n T_n \left( \frac{x}{c_r} \right)
$$

where $T_n$ is the shifted Chebyshev polynomial of order $n$. Note that the same mode coefficients $b_n$ are used for the potential jump and pressure jump series expansions. The series expansions for pressure satisfies the $x^{-1/2}$ type singularity at the leading edge as required for subsonic relative flows.

Substituting equations (11) and (12) into the linearized Bernoulli's equation (3), one can show that the $n$th potential mode function is simply related to the pressure mode function by the following equation:

$$
H_n \left( \frac{x}{c_r} \right) = e^{-i\omega t_{(n/\pi)}} \sum_{j=0}^{N_o} d_j^p \int_0^{x/c_r} (1-z)^{1/2} z^{j-1/2} e^{i\omega t_{(n/\pi)}} dz
$$

where $d_j^p$ are the coefficients of polynomial expansion of the shifted Chebyshev polynomial $T_n^*(x)$, i.e.,

$$
T_n^*(x) = \sum_{j=0}^{N_o} d_j^p x^j
$$

Note that the pressure mode functions are real and the potential mode functions are frequency dependent and complex.

Representative pressure mode functions are shown in Figure 3 from the first six modes, and the corresponding potential mode functions are shown in Figures 4, 5 and 6. Note that all
pressure mode functions vanish at the trailing edge \((x = 1)\) and all potential mode functions vanish at the leading edge \((x = 0)\). See Chi (1990d) for the analytical details.

By choosing the number of modes \(N_{\text{mode}}\) equal to the number of chordwise panels for each radial station and applying equation (11) to the centroids of the blade panels, one would obtain a linear transformation between the unknown potential jumps and the mode coefficients \(b_o\). Then, by assuming constant \(\Delta \phi\) panels in equation (4), one obtains the following set of simultaneous algebraic equations in which the unknowns are a collection of mode coefficients \(\{b_o\}\) for all chordwise strips:

\[
[A]\{b_o\} = \left[ \begin{array}{c} w \end{array} \right]
\]

(13)

where the elements of the aerodynamic influence matrix are simply the integrals of the potential kernel function over the reference blade and wake surfaces.

Here, the number of unknowns is the same as the number of blade elements which is equal to the sum of the numbers of mode coefficients from all chordwise strips. Then, selecting the centroids of the blade elements to match the upwash velocity on the right hand side of equation (13) will yield the solution for the mode coefficients. The jumps in perturbation pressure are then calculated using equation (12).

**Aerodynamic Influence Matrix**

The calculation of the aerodynamic influence matrix \([A]\) in equation (13) must be carefully handled. This is primarily due to the singular nature of the kernel function \(K(r_o, \vec{r})\) as \(r_o\) approaches \(\vec{r}\). The singular part of the kernel function must be integrated analytically. Besides, for certain combinations of duct sizes and flow parameters, nondecaying propagating waves will exist and the kernel function would oscillate with a finite amplitude for large \(z_o - z\) values. These propagating waves must be integrated analytically over the semi-infinite wake region extending to downstream infinity.

Therefore, for a given upwash point, the surface integral given by Eq. (4) is separated into three parts: one associated with the integral over the blade panel surface whose centroid is the upwash point \(W_{\text{blade singular}}\), one associated with integrals covering the rest of the blade panels \(W_{\text{blade regular}}\), and one for the entire wake region \(W_{\text{wake}}\). Therefore we have

\[
W(\vec{r}) = W_{\text{blade singular}}(\vec{r}) + W_{\text{blade regular}}(\vec{r}) + W_{\text{wake}}(\vec{r})
\]

**Integral over Upwash–Point panel, \(W_{\text{blade singular}}\)** At the centroid of the upwash panel, the upwash velocity is matched exactly. The portion of the integral shown in equation (4) that covers the upwash panel element, i.e. \(W_{\text{blade singular}}\) must be evaluated in two parts by writing the kernel function \(K(r_o, \vec{r})\) as

\[
K(r_o, \vec{r}) = K_s(r_o, \vec{r})(F) + [K(r_o, \vec{r}) - K_s(r_o, \vec{r})]
\]

where \(K_s(r_o, \vec{r})\) is the singular term given by Eq. (9). The first term \(K_s\) is integrated analytically as follows:

\[
\int \int K_s(r_o, \vec{r}) \Delta \phi (r_o) dS_o = \int \int K_s(r_o, \vec{r}) dS_o
\]

upwash panel element

and the second term \(K - K_s\) is integrated numerically using Gaussian quadrature.

**Integral Over Blade Panels Besides Upwash–Point Panel, \(W_{\text{blade regular}}(\vec{r})\)** The kernel function is regular outside the upwash point panel, and therefore the integral \(W_{\text{blade regular}}(\vec{r})\) is evaluated using Gaussian quadrature.

**Integral Over Wake, \(W_{\text{wake}}(\vec{r})\)** The integral over the wake surface \(W_{\text{wake}}(\vec{r})\) requires a special treatment because (a) the wake potential jump is related to the trailing edge potential jump by a simple exponential factor, (b) the kernel function \(K(r_o, \vec{r})\) in general oscillates indefinitely toward downstream infinity \((z_o - z \to \infty)\). In the wake integral,

\[
W_{\text{wake}}(\vec{r}) = \int \int K(r_o, \vec{r}) \Delta \phi (r_o) dS_o
\]

wake

the potential jump in the wake \(\Delta \phi\) is related to the potential jump at the trailing edge traced back along the same helix for a fixed radius \(r\), i.e. Eq. (10). The kernel function \(K(r_o, \vec{r})\) oscillates indefinitely toward downstream infinity with constant amplitudes and a finite number of spatial periods. Representing
the spatially oscillating part of the kernel function by $K_\omega$, one can write the full kernel as

$$K(\vec{r}_0, \vec{r}) = K_\omega(\vec{r}_0, \vec{r}) + [K(\vec{r}_0, \vec{r}) - K_\omega(\vec{r}_0, \vec{r})]$$

Then the wake integral becomes

$$W_{\text{wake}}(\vec{r}) = I_\omega(\vec{r}) + \Delta I(\vec{r})$$

where

$$I_\omega(\vec{r}) = \int \int K_\omega(\vec{r}_0, \vec{r}) \Delta \phi(\vec{r}_0) \, dS_0$$

wake

$$\Delta I(\vec{r}) = \int \int [K(\vec{r}_0, \vec{r}) - K_\omega(\vec{r}_0, \vec{r})] \Delta \phi(\vec{r}_0) \, dS_0$$

with the area element on a helical surface given by

$$dS_0 = \left\{ \left( \frac{U}{\Omega} \right)^2 + \vec{r}_0^2 \left[ 1 + \left( \frac{d\theta}{d\xi_0} \right)^2 \right] \right\}^{1/2} \, dr_0 \, d\theta_0$$

The integral $\Delta I(\vec{r})$ is evaluated using Gaussian quadrature. Meanwhile, the unknown $\Delta \phi(x)$ for a wake panel at a given radius is replaced by the new unknown $\Delta \phi(x_{TE})$ for the trailing edge panel at the same radius using Eq. (10).

The integral $I_\omega$ must be evaluated analytically. Because of the use of the helical coordinate $x$ in representing the wake potential jump in Eq. (10), it is convenient to choose $r_0, x_0$ instead of $r_0, \theta_0$ as the independent variables in evaluating the surface integral $I_\omega$, i.e.

$$I_\omega(\vec{r}) = \int_{r_0}^{R_f} \Delta \phi(x_{TE}(r_0)) \, e^{ix_{TE}(r_0)} \left\{ \left( \frac{U}{\Omega} \right)^2 + \vec{r}_0^2 \left[ 1 + \left( \frac{d\theta}{d\xi_0} \right)^2 \right] \right\}^{1/2} \, dr_0$$

Here, the integral is first carried out along each helix ($x_0$ integration) followed by an integral in the radial direction ($r_0$ integration). The radial integral is carried out again using the Gaussian quadrature algorithm after the $x_0$ integral inside the square bracket is evaluated analytically.

The oscillating part of the kernel function consists of a finite number of terms of the following form:

$$e^{i\alpha x_0}$$

where $\alpha$ is a real number. Using this asymptotic form in the square bracket term involving $K_\omega(\vec{r}, \vec{r})$, one needs to evaluate integrals of the following type:

$$\int_{x_{TE}(r_0)}^{x_{TE}(R_f)} e^{-i\alpha x_0} \, dx_0$$

which has a finite value given by

$$\frac{-e^{-i\alpha x_0}}{i(-\omega + a)}$$

by assuming that $\omega$ has a small but negative imaginary part. The assumption of a small but negative imaginary part of the frequency $\omega$ is a means of introducing artificial viscosity into the flow model. An assumption of a positive imaginary part of $\omega$ would have made the integral along the helix become infinitely large, and this would correspond to "negative viscosity" which is physically unrealizable.

Therefore, the $K_\omega$ integral can be written as

$$\int_{x_{TE}(r_0)}^{x_{TE}(R_f)} K_\omega(\vec{r}_0, \vec{r}) e^{i\alpha x_0} \, dx_0$$

$$= \sum_j \frac{i e^{i\alpha x_{TE}(r_0)}}{\alpha x_{TE}(r_0) - \alpha x_{TE}(r_0)}$$

where the summation over $j$ includes as many terms as the flow condition dictates.

RESULTS

Baseline Cascade Configurations

Three cascade configurations, designated as Cascade A, Cascade B, and Cascade C, are analyzed using the analysis procedures described above. The number of panels for blade discretization depends on the size of the blade and the required accuracy for the unsteady pressure jump. Generally, satisfactory results can be obtained using eight chordwise elements, six spanwise elements, and a wake axial length equal to the maximum axial extent of the blade. A typical computational runtime is thirty minutes on a VAX workstation Model 3100.

The geometric properties and flow conditions for these cascade configurations are given in Table 1. The helix numbers $\sigma(r)$ for all these baseline cascade configurations are zero so that the helical blade for these cascades always contains a straight radial line which defines the radial coordinate. All pressure jump data are normalized by the local values of $\rho U_2^2$.

**Cascade A**. Cascade A has thirty high aspect-ratio blades (aspect ratio near 5) with a constant axial chord. The midchord locus of the blade coincides with a straight radial line corresponding to zero helix number $\sigma(r) = 0$. 

7
For an axial flow Mach number $M = 0.5$ and a nondimensional rotor speed $\Omega/U = 1$, Lordi and Homicz (1981) calculated the sectional lift forces for a steady flow produced by a unit angle-of-attack twist of the blade. For the same flow condition, Figure 7 shows the detailed pressure jump distributions calculated using the present three dimensional theory and the classical two dimensional strip theory (Chi, 1980). In the three dimensional analysis, the blade is discretized into five strips of equal radial width and each strip is further divided into four equal chordwise elements. For each radial station, the length of the wake surface included in the analysis is equal to the blade chord for that radial station. Because of the high aspect ratio of the blade and the constancy of the blade displacement along the span, the steady pressure distributions for all radial stations exhibit very strong two dimensional flow nature.

**Cascade B.** Cascade B represents a rotor of thirty high aspect-ratio blades (aspect ratio near 6) with a constant axial chord which is one-third smaller than the Cascade A blade. The midchord locus is again radially straight. For an axial flow Mach number 0.3 and a nondimensional rotor speed $\Omega/U$ equal to 2.4744, Namba and Ishikawa (1983) calculated the pressure jump distributions due to the six nodal-diameter circumferential bending vibration of forward traveling wave type at a reduced frequency $\omega/U$ equal to 3.0. The first bending mode shape of a cantilever beam was used to represent the radial variation of the bending displacement amplitude. However, the displacement amplitude at the blade tip was not clearly stated in his paper.

Assuming the amplitude of the circumferential bending displacement at the blade tip is unity relative to the fan case radius, the three dimensional unsteady pressure distributions are calculated using six aerodynamic strips with four chordwise blade elements and twenty wake elements for each strip. In Figure 8, the three dimensional results are compared with the two dimensional strip theory. Because of the large variation of bending displacement amplitude from the blade root section to the blade tip section, a strong radial flow interaction apparently brings down the two dimensional strip theory load near the blade tip by more than fifty percent. On the other hand, near the hub, the unsteady load is increased from its minimum two dimensional load ( exactly zero at the hub because zero local blade motion) to a moderate level.

**Cascade C.** Cascade C represents a fan rotor of fifteen blades with an aspect ratio near 2. The blade chord remains constant ( chord normalized by fan case radius equal to 0.33333) along the span so that the axial chord decreases radially outward. The leading edge locus is a straight radial line which is also the pitching axis of the blade. This cascade configuration was used by Williams (1990) in his low Mach number propfan airload calculations.

For an axial flow Mach number 0.5 and a nondimensional rotor speed 1.597, Figure 9 shows the calculated three dimensional pressure jump distributions compared to the two dimensional strip theory load due to a zero nodal-diameter pitching oscillation about the straight leading edge locus at a reduced frequency $\omega/U$ equal to 2.396. In the three dimensional analysis, the blade span is divided into six strips and each strip is further discretized into eight chordwise elements. The wake length included is twice the blade chord at each radial station. The strong two dimensional nature of the unsteady pressure is clearly seen primarily because of the pitching amplitude is constant along the blade span.

With the same flow condition and pitching oscillation frequency, calculation is made by introducing a linear pitching amplitude distribution along the span (zero pitching at the root and unit pitching angle at the tip). The result is shown in Figure 10. The pitching airload is reduced near the tip and the hub load is increased because of the enhancement of the three dimensional radial flow interaction by the significant radial variation of the blade displacement amplitude.

**Effect of Bending, Bowing and Helical Sweep**

To study the effect of blade bending (in the engine axis direction), bowing (in the circumferential direction), and sweeping (by sliding blade sections along the helix at each radial station), the Cascade C baseline blade configuration is bent axially, bowed (i.e., bent circumferentially), and swept and three dimensional unsteady aerodynamic analyses are carried out. Any of these blade deformation will introduce a nonzero helix number distribution $\sigma(t)$ along the blade span. As the blades are deformed, the pitching axis for each radial station still remains at its own leading edge in order to make a sensible airload comparison.

**Axial Bending.** In Figure 11, the three dimensional pressure jump distributions for the baseline Cascade C rotor are compared with those for an axially bent rotor. The amount of axial bending is distributed linearly along the blade span with zero bending at the blade root and the maximum axial backward bending distance equal to ten percent of the fan case radius. The majority of the unsteady airload distribution pattern shows an increase in its magnitude due to backward bending of the blades.

**Bowing (I.e., Circumferential Bending).** Figure 12 shows the comparison between the pressure jump distributions for the baseline Cascade C rotor and those for a circumferentially bowed rotor. The amount of circumferential bending is distributed linearly along the blade span with zero bowing at the blade root and a maximum circumferential angle movement of 9.15 degrees opposite to the direction of rotation at the blade tip. The airloads are seen to increase for the outer span and decrease for the inner span due to the circumferential shift of the blades.

**Sweep.** The baseline Cascade C blades are swept by sliding the helical chord along the helix forward or backward at each radius. Blade sweeping defined in this manner maintains the
zero helix number associated with the baseline blades and simply changes the leading edge and trailing edge locations of each blade section along the helix.

For blades with backward sweep, Figures 13 and 14 show the real and imaginary parts of the pressure jump distributions respectively. Two swept configurations are analyzed: one corresponds to an axial shift of 30 percent of the fan case radius, and another 50 percent of the fan case radius. Backward sweep is seen to reduce the airload in general. For blades with forward sweep, similar results are shown in Figures 15 and 16.

CONCLUSIONS

An analytical method has been developed to compute subsonic unsteady pressure jump distributions on oscillating blades of an ducted engine fan. The method utilizes an aerodynamic kernel function that absorbed the fan hub and case boundary conditions and the interblade influences so that an integral equation that relates the potential jump across the lifting surface to the upwash distribution can be solved by discretizing only the blade and wake surfaces.

Representative results calculated for three standard cascade configurations are given covering zero to high frequency oscillations due to bending as well as torsional motion. One of the baseline cascade configuration is further deformed to allow the effect of blade axial bending, bowing (i.e., circumferential bending), and sweeping on unsteady airloads be clearly observed.

The three dimensional effect is generally important for low aspect ratio, swept blades for obvious reasons. For high aspect ratio blades, the three dimensional effect is also important, because the radial variation of the vibration mode shape is in general large enough to trigger significant spanwise unsteady flow interactions that usually reduce the airload for the outer span and elevate the airload for the inner span sections.

REFERENCES


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<table>
<thead>
<tr>
<th>TABLE 1</th>
<th>BASELINE CASCADE PROPERTIES</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Cascade A</td>
</tr>
<tr>
<td>Axial Mach Number (M)</td>
<td>0.5</td>
</tr>
<tr>
<td>Rotor Speed (U/Ω)</td>
<td>1</td>
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<tr>
<td>Hub-to-Tip Ratio (R_{h}/R_{T})</td>
<td>0.5</td>
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<tr>
<td>Number of Blades (N_{b})</td>
<td>30</td>
</tr>
<tr>
<td>Special Feature</td>
<td>Midchord Locus is Radial</td>
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<tr>
<td>Axial Chord (C_{a}/R_{T})</td>
<td>Constant 0.1</td>
</tr>
<tr>
<td>Helical Chord (C_{r}/R_{T})</td>
<td>Varying Along Span</td>
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<tr>
<td>Blade Motion</td>
<td>Unit Angle of Attack</td>
</tr>
<tr>
<td>Frequency (ω/Ω)</td>
<td>0</td>
</tr>
<tr>
<td>Nodal Diameter (N_{a})</td>
<td>0</td>
</tr>
</tbody>
</table>

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**Isolated airfoil**

\[
\text{Velocity vector} \quad \mathbf{q} = \mathbf{U} + \mathbf{q}
\]

**Cascade blades**

\[
\phi = Ux + \mathbf{U} \times \mathbf{r}
\]

**Absolute velocity potential**

\[
\phi = 2 \pi n
\]

**Fig. 1** Small Perturbation Concept.

**Fig. 2** Helical Blade Section.
Fig. 3  Pressure Mode Functions $\sqrt{1-x/c\over x/c} T_n(x/c)$ for $n = 0, 1, 2, 3, 4, 5$

Fig. 4  Potential Mode Functions $H_n(x/c)$ versus $x/c$ for $n = 0, 1, 2$ assuming $\omega_c/U_r = 1$
Fig. 5 Potential Mode Functions $H_n(x/c)$ versus $x/c$ for $n = 3$ and 4 assuming $\omega_c/U_r = 1$

Fig. 6 Potential Mode Functions $H_n(x/c)$ versus $x/c$ for $n = 5$ and 6 assuming $\omega_c/U_r = 1$
2D Strip Theory

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3D Helical Surface Theory

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Fig. 7 Cascade-A Steady Pressure Jump Distributions for Unit Angle of Attack: 2D versus 3D

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2D Strip Theory - Real Part

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2D Strip Theory - Imaginary Part

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3D Theory - Real Part

---

3D Theory - Imaginary Part

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Fig. 8 Cascade-B Unsteady Pressure Jump Distributions for Bending Vibration: 2D versus 3D
Fig. 9 Cascade–C Unsteady Pressure Jump Distributions for Unit Angle of Attack
Pitching Oscillation about Leading Edge: 2D versus 3D

Fig. 10 Cascade–C Unsteady Pressure Jump Distributions for Linear Pitching Oscillation:
2D versus 3D
Fig. 11 Effect of Axial Bending of Cascade-C Blades on 3D Unsteady Pressure Jump Distributions

Fig. 12 Effect of Tangential Bowing of Cascade-C Blades on 3D Unsteady Pressure Jump Distributions
Fig. 13 Effect of Linear Backward Sweep of Cascade-C Blades on Real Part of 3D Unsteady Pressure Jump Distributions.

Fig. 14 Effect of Linear Backward Sweep of Cascade-C Blades on Imaginary Part of 3D Unsteady Pressure Distributions.
Fig. 15 Effect of Linear Forward Sweep of Cascade–C Blades on Real Part of 3D Unsteady Pressure Distributions.

Fig. 16 Effect of Linear Forward Sweep of Cascade–C Blades on Imaginary Part of 3D Unsteady Pressure Distributions.