A FINITE ELEMENT APPROACH TO THE ANALYSIS OF ROTATING BLADED-DISK ASSEMBLIES COUPLED WITH FLEXIBLE SHAFT*

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ABSTRACT

A method for dynamic analysis of flexible bladed-disk/shaft coupled systems is presented in this paper. Being independent substructures first, the rigid-disk/shaft and each of the bladed-disk assemblies are analyzed separately in a centrifugal force field by means of the finite element method. Then through a modal synthesis approach the equation of motion for the integral system is derived. In the vibration analysis of the rotating bladed-disk substructure, the geometrically nonlinear deformation is taken into account and the rotationally periodic symmetry is utilized to condense the degrees of freedom into one sector. The final equation of motion for the coupled system involves the degrees of freedom of the shaft and those of only one sector of each of the bladed-disks, thereby reducing the computer storage. Some computational and experimental results are given.

INTRODUCTION

An analytical approach and the corresponding software package, to be practical in industry, are developed to study the dynamics of rotating bladed-disk/shaft coupled systems.

In the past, when investigating the dynamic property of the bladed-disk/shaft subsystem of aeroengines, the flexible bladed-disk and the rotating shaft were analyzed separately, which means that the coupling effects between these two substructures were neglected. Such a method is reasonable in the case where the fundamental natural frequency of the bladed-disk is much higher than that of the rotor system or, in other words, the rigidity of the bladed-disk is much larger than that of the shaft. However, the modern trend of lighter structure and higher speed in aeroengine design increases the elastic coupling effect between the bladed-disk and the shaft. The dynamic analysis of bladed-disk/shaft coupled system is therefore of practical interest in recent years (Crandall and Dugundi, 1981, Kharder and Lowey, 1990, Lowey and Karder, 1984, Crawley et al., 1986, Wu and Flowers, 1992).

The main difficulties of our problem consist of the limitation of computer storage capacity and the exact evaluation of the stiffness effect of centrifugal force. A reliable FEM model of a realistic twisted blade would contain hundreds of DOF. A bladed-disk with tens of blades therefore has about 10^4 DOF. The number of DOF of the FEM model of a rotor system with multiple bladed-disk assemblies might be 10^5 in order of magnitude which is too large to be analyzed by most computers. Thus, in most literature up to now (Kharder and Lowey, 1990, Lowey and Karder, 1984, Crawley et al., 1986), the blade was roughly modeled by beam, strip and plate to avoid the storage capacity difficulty. In fact, it has been already gradually aware of that in order to evaluate the stiffening effect of the centrifugal force correctly, one should execute the geometrically nonlinear analyses on bladed-disk assemblies (Agrawal, 1991, Leung, 1990, Bloch and Ryan, 1989). However, the necessity of the nonlinear analysis has not been rec-

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ognized in many of the earlier works. The error in stiffness matrix may even lead to the incorrect prediction of the instability behavior of rotor system.

Our approach and results are as follows:
1. A 3-dimensional geometrically nonlinear finite element analysis of bladed-disk substructure is carried out. The dynamic equilibrium configuration of the bladed-disk in the centrifugal force field and the corresponding tangential stiffness matrix at this equilibrium position are obtained exactly.
2. Taking advantage of the rotationally periodic symmetry of the bladed-disk, the CN group theory algorithm is used to condense the degrees of freedom of the substructure into one sector (1/N part of the bladed-disk assembly) with only 1/N DOF of the original problem, thereby reducing the computer storage.
3. After analyzing the rigid-disk/shaft and each of the bladed-disk assemblies as substructures, using a modal synthesis technique the general equation of motion for the coupled system and the corresponding gyroscopic eigenvalue problem are obtained. The final equation of motion involves the degrees of freedom of the shaft and those of only one sector of each of the bladed-disk assemblies.
4. According to the above theory, a large scale FEM software package DASOBD is composed. The software can be used to analyze practical industrial rotor systems.
5. Some computational and experimental results are given. The agreements between the numerical and experimental investigations show the validity and the effectiveness of both the theory and the software.

AN OUTLINE OF GEOMETRICALLY NONLINEAR DYNAMIC ANALYSIS OF A ROTATING FLEXIBLE BODY

General Equation of Motion and the Dynamic Equilibrium Configuration

Set a coordinate system O-xyz rotating with the same spinning velocity of the body. Let r and P denote respectively the non-deformed position and the deformation of an arbitrary point of the body relative to the rotational coordinate system:

\[
r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad P(x, y, z) = \begin{bmatrix} u \\ v \\ w \end{bmatrix}
\]  

Let us consider the finite element model of the body. The continuous deformation can be expressed as

\[
P(x, y, z) = N(x, y, z)\delta
\]  

where \(\delta\) and N are, respectively, the generalized coordinate set and the shape function matrix. The kinetic energy of the body in terms of \(\delta\) and \(\delta\) then is

\[
T = \frac{1}{2}\delta^T M \delta + \frac{1}{2}\delta^T G \delta + \frac{1}{2}\delta^T K_\Omega \delta + \delta^T A + \delta^T B + T_0
\]

where

\[
M = \int_0 \rho N^T N dv, \quad G = 2 \int \rho N^T \Omega N dv
\]

\[
K_\Omega = \int \rho N^T \Omega^T \Omega N dv, \quad A = \int \rho N^T \Omega^T \Omega \theta dv
\]

\[
\Omega = \begin{bmatrix}
\Omega_1 & \Omega_2 & \Omega_3
\end{bmatrix}
\]

\[
\delta^T(N_x^T N_x) \delta^T(N_x^T N_y) \delta^T(N_y^T N_z)
\]

\[
B_N(\delta) = \frac{1}{2}
\]

\[
D = \begin{bmatrix}
\delta^T(N_x^T N_x) + \delta^T(N_y^T N_y) + \delta^T(N_z^T N_z)
\delta^T(N_x^T N_x) + \delta^T(N_y^T N_y) + \delta^T(N_z^T N_z)
\delta^T(N_x^T N_x) + \delta^T(N_y^T N_y) + \delta^T(N_z^T N_z)
\end{bmatrix}
\]

D is the generalized Hooke's elasticity matrix. The nonlinear strain energy then is such that

\[
V = \frac{1}{2} \int_0 \varepsilon^T \varepsilon dv = \frac{1}{2} \int_0 \varepsilon^T D \varepsilon dv
\]

From (3) and (9), we obtain the general equation of motion as

\[
M \ddot{\delta} + G \dot{\delta} + \psi(\delta) - K_\Omega \delta = A
\]

where

\[
\psi(\delta) = \frac{\partial V}{\partial \delta} = \int (B^T + B^T_N) \sigma dv
\]

\[
\text{Set a coordinate system } O-xyz \text{ rotating with the same spinning velocity of the body. Let } r \text{ and } P \text{ denote respectively the non-deformed position and the deformation of an arbitrary point of the body relative to the rotational coordinate system:}
\]

\[
r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad P(x, y, z) = \begin{bmatrix} u \\ v \\ w \end{bmatrix}
\]  

\[
P(x, y, z) = N(x, y, z)\delta
\]  

\[
\text{where } \delta \text{ and } N \text{ are, respectively, the generalized coordinate set and the shape function matrix. The kinetic energy of the body in terms of } \delta \text{ and } \delta \text{ then is}
\]

\[
T = \frac{1}{2}\delta^T M \delta + \frac{1}{2}\delta^T G \delta + \frac{1}{2}\delta^T K_\Omega \delta + \delta^T A + \delta^T B + T_0
\]
The dynamic equilibrium configuration $\delta_0$ is next extracted. Taking $\delta_0 = \delta_0 = 0$, from (10) we have

$$\psi(\delta_0) - K_\Omega \delta_0 - A = 0$$

(12) is a nonlinear equation which can be solved through Newton-Raphson’s iterative approach.

Equation of Perturbed Motion and Tangential Stiffness Matrix

Inserting $\delta = \delta_0 + q(t)$ into (10), here $q(t)$ denotes the perturbed motion, with the first order quantities remained we obtain

$$M \ddot{q} + G\dot{q} + K_T(\delta_0)q = 0$$

where

$$K_T(\delta_0) = \left. \frac{\partial^2 \psi}{\partial \delta^2} \right|_{\delta_0} - K_\Omega$$

$$= K_0 + K_L(\delta_0) + K_\sigma(\delta_0) - K_\Omega$$

(14)

$$K_0 = \int_v B_L^T DB_L dv$$

$$K_L(\delta_0) = \int_v (2B_L^T DB_N + 2B_N^T DB_L$$

$$+ 4B_N^T DB_N) dv$$

(15)

$$K_\sigma(\delta_0) = \int_v \bar{H} \bar{\delta}_0 Hdv, \quad H = \begin{bmatrix} N_x & \cdots & \cdots & N_x \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ N_y & \cdots & \cdots & N_y \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ N_z & \cdots & \cdots & N_z \end{bmatrix}$$

(16)

$K_0$ is the linear stiffness matrix, whereas $K_\sigma$ and $K_L$ are called, respectively, the nonlinear geometrical stiffness matrix and the the large displacement stiffness matrix.

For most bladed-disk assemblies, we have $\dot{\Omega} \Omega P \approx 0$, thus the gyroscopic term can be neglected and then the equation of perturbed motion (13) is simplified into

$$M_d \ddot{q}_d + K_d q_d = 0$$

(17) is decoupled into the following $[(N - 1)/2]$ independent $n$-order Hermitian generalized eigenproblems

$$(B_r - \omega^2 A_r)z_r = 0, \quad r = 1, \ldots, [(N - 1)/2]$$

(23) and the following real symmetric generalized eigenproblems

$$(B_0 - \omega^2 A_0)z_0 = 0, \quad (B_{N/2} - \omega^2 A_{N/2})z_{N/2} = 0$$

(24) where

$$V^* M_d V = block-diag\{A_r\}$$

$$V^* K_d V = block-diag\{B_r\}$$

(25)

Letting the $r$-nodal diameter complex normal eigenvector set of (23) to be

$$z_r = [z_{r1}, -iz_{r1}, z_{r2}, -iz_{r2}, \cdots, z_{rn}, -iz_{rn}]$$

(26)
then, from (22) we obtain the corresponding r-nodal diameter physical modes of \( S_j \) and \( \tilde{S}_j \) as follows:

\[
X_{r_j} = \sqrt{\frac{2}{N}} \text{Re}(z_r)[I_{2n} \cos r(j - 1)\alpha] - \tilde{I}_{2n} \sin r(j - 1)\alpha \tag{27}
\]

\[
X_{\tilde{r}_j} = \sqrt{\frac{2}{N}} \text{Re}(\tilde{z}_r)[I_{2n} \cos r(j - 1)\alpha] - \tilde{I}_{2n} \sin r(j - 1)\alpha \tag{28}
\]

where

\[
\tilde{I}_{2n} = \text{diag}(\tilde{I}_2), \quad \tilde{I}_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}
\]

\[
Z_r = \begin{bmatrix} I_{\omega} & 0 \\ 0 & I_{\tilde{\omega}} \end{bmatrix} z_r \tag{29}
\]

**ANALYSIS OF RIGID-DISK/SHAFT SUBSTRUCTURE**

Suppose the generalized coordinate set of the FEM model of the shaft to be \( q_s \). The kinetic and strain energy and the equation of motion are (Zhang, 1990):

\[
T = \frac{1}{2} q_s^T M_s q_s + \frac{1}{2} q_s^T G_s q_s, \quad V = \frac{1}{2} q_s^T K_s q_s \tag{30}
\]

\[
M_s q_s + \Omega G_s q_s + K_s q_s = 0 \tag{31}
\]

Taking \( \Omega = 0 \), we obtain a vibration eigenproblem

\[
(K_s - \omega^2 M_s)q_s = 0 \tag{32}
\]

From (32) we can extract \( 2n_s \) double-repeated eigenroots \( \omega^2 \) and the corresponding orthogonal modes \( \phi_{2j-1} \) and \( \phi_{2j} \), here \( n_s \) is the number of nodes of the shaft FEM model. The total modal set then is

\[
\Phi = [\phi_1, \phi_2, \ldots, \phi_{4n_s-1}, \phi_{4n_s}] \tag{33}
\]

**MODAL SYNTHESIS APPROACH FOR BLADED-DISK/SHAFT COUPLED SYSTEM Modal Coordinate Sets**

In our bladed-disk/shaft coupled system, the whirl motion of the shaft can excite the consistent rigid-body motion as well as the flexible vibration of the bladed-disk assemblies. However, it has been proved (Hu, 1987) that in such cases only the single-nodal diameter vibration modes will be excited. According to the principle of modal synthesis approach, we should take the first \( 2t(t \ll n_s) \) modes \( \Phi_t = [\phi_1, \phi_2, \ldots, \phi_{2t-1}, \phi_{2t}] \) of (33) and the first \( 2s(s \ll n) \) single-nodal diameter modes \( \Phi_1 \) of (28) as the two component modal sub-spaces to describe the real deformation of the two substructures

\[
q_s = \Phi_1 q_1 \tag{34}
\]

\[
q_2 = \Phi_1, R_1, q_2 = \sqrt{2/N} \text{Re}(Z_1)[I_{2s} \cos (k - 1)\alpha] - \tilde{I}_{2s} \sin (k - 1)\alpha] R_1, q_2 \tag{35}
\]

where

\[
R_1 = \text{diag} \{ R_{11} \}, \quad R_{11} = \begin{bmatrix} \cos \Omega t & \sin \Omega t \\ -\sin \Omega t & \cos \Omega t \end{bmatrix} \tag{36}
\]

\( q_1 \) and \( q_2 \) are the generalized coordinate sets of the shaft and the bladed-disk respectively. For multiple bladed-disks, \( q_2 \) in (35) represents the total set of the generalized coordinates of the bladed-disks.

**Equation of Motion for Coupled System**

After a series of coordinate transformations and a rather lengthy deduction, the kinetic and strain energy of the bladed-disk/shaft system relative to the modal coordinates can be obtained as

\[
T = \frac{1}{2} q^{T} M q + \Omega q^{T} H q + \frac{\Omega^2}{2} q^{T} M q \tag{37}
\]

where

\[
q_1 \quad M = \begin{bmatrix} I_{2t} & \frac{1}{2} \Phi_1^T E^T \left[ A_2 \right] \\ 0 & \text{Sym.} \end{bmatrix}
\]

\[
H = \begin{bmatrix} \frac{1}{2} \Phi_1^T G_1 \Phi_1 & \Phi_1^T E^T \left[ A_4 - A_3 I_{2s} \right] \\ \Phi_1^T G_1^T \Phi_1 & \text{Sym.} \end{bmatrix} \tag{38}
\]

\[
K = \begin{bmatrix} \Lambda_s & \Lambda_{14} \\ \Lambda_{14} & \text{Sym.} \end{bmatrix}
\]

\[
A_1 = I_{2s} + 2\tilde{I}_{2s} A_1, \quad A_{j} = I_{2s} + \tilde{I}_{2s} A_{j}, \quad (j = 2, 3, 4)
\]

\[
\Pi_1 = Re \left( Z_1 \right) \left( \int_{S_1} \rho N^T R_1 N dv \right) Re(Z_1)
\]

\[
\Pi_j = \sqrt{\frac{N}{2}} \left( \int_{S_1} \rho R_j N dv \right) Re(Z_1), \quad (j = 2, 3, 4)
\]
\[ R_1 = \begin{bmatrix} 0 & \mathbf{I}_2 \\ \mathbf{0} & \mathbf{I}_2 \end{bmatrix}, \quad R_2 = [0, \mathbf{I}_2] \]

\[ R_3 = \begin{bmatrix} \zeta & 0 & -\xi \\ -\eta & \xi & 0 \\ -\xi & \zeta & 0 \end{bmatrix}, \quad R_4 = -\begin{bmatrix} \eta & \xi & 0 \\ \zeta & 0 & \xi \end{bmatrix} \]

where

\[ G = H - H^T \]  

The corresponding generalized gyroscopic eigenproblem of (39) is

\[ (-\omega^2 \mathbf{M} + i\omega \mathbf{QG} + \mathbf{K} - \Omega^2 \mathbf{M}_e)\mathbf{q} = 0 \]  

Many algorithms have been developed for solving (41) if \( \mathbf{K} - \Omega^2 \mathbf{M}_e \) is positive definite. The method proposed by Meirovitch (Meirovitch, 1974) is to transform a \( n \)-order quadratic gyroscopic eigenproblem (41) into a \( 2n \)-order symmetric eigenproblem. Recently, the authors developed a new algorithm (Zhang and Zhang, 1990) which reduces (41) to two \( n \)-order symmetric
eigenproblems, therefore the computer storage and the computational time are significantly diminished. This method is much effective for large scale gyroscopic eigenproblem arising from engineering.

**COMPUTATIONAL AND EXPERIMENTAL EXAMPLES**

According to the above theory, a large scale FEM software package DASOBD is developed for analyzing bladed-disk/shaft coupled system of industrial rotating machinery. Fig.1 shows the flow chart of DASOBD.

Some laboratorial experiments and computational analyses have been studied to verify the validity and efficiency of the theory as well as the software.

A thin plate-like model with 8-order periodic symmetry, mounted at the 1/4 length of a simple-supported shaft, was used to simulate the bladed-disk assembly. The scheme of the laboratorial rotor is shown in Fig.2. The critical rotating speeds of the rotor were measured. The equipment is shown in Fig.3. The rotating speed was first increased from 0 to 4300 r/min and then decreased. In the speed decreasing period the first critical speed was measured as 2730 r/min (45.5 Hz). When the rotating speed was higher than 8800 r/min, the vibrating amplitude of the rotor increased gradually, then one of the blades broke at 11000 r/min and the shaft appeared as the second bending mode. Therefore the experimental value of the second critical rotating speed should be about 11000 r/min.

The whirl frequency curves $\omega_i = \omega_i(\Omega)(i = 1, 2, \cdots, 8)$ of the laboratorial rotor were computed. Fig.4 is the campbell diagram, where the dashed and solid lines represent the backward and forward whirl frequency curves respectively. The intersection points of forward whirl frequency curves and line $\omega = \Omega$, 2767 r/min (46.1 Hz) and 11115 r/min, are the computed critical speeds which are in good agreements with the experimental values.

To assess the coupling effect, the campbell diagrams calculated of both the rigid-disk/shaft and the rotating bladed-disk are shown in Fig.5 and Fig.6. When using the rigid-disk/shaft model, the computed first and second critical speeds are 2861 r/min and 11238 r/min. Clearly, the results given by the bladed-disk/shaft coupled model are much more accurate. On the other hand, it can be observed from the plots that in the coupled case the flexible disk frequencies are significantly depressed due to the shaft flexibility, whereas those flexible shaft frequencies are comparatively less affected. Comparing resonance points from the uncou-
pled cases to the coupled one shows the substantial effect produced by coupling. For example, without shaft flexibility, the resonance of the third flexible disk mode with the second harmonic and that of the fourth flexible disk mode with the third harmonic are well beyond 12000r/min. In the coupled system, however, the $2\Omega$ resonance of the third disk mode and the $3\Omega$ resonance of the fourth disk mode are, respectively, 11162r/min and 9009r/min. Therefore, if one is to obtain a reliable dynamic prediction of a rotating bladed-disk/shaft system, an integral analysis is undoubtedly necessary.

This research has been applied to realistic engine design. The lower pressure turbine of a turbofan engine was analyzed and the results are satisfactory.

REFERENCES


