ABSTRACT

The vibration of grouped blades on a flexible disk should, for purposes of economy and clarity of modal identification, be analyzed using procedures developed for cyclically-symmetric structures. In this paper, a numerical model, based on the theory of cyclically-symmetric structures, is applied to the vibration analysis, and in particular, the harmonic response, of a flexible disk supporting a number of groups, or packets, of turbine blades. Results are presented to show variations in the modal participation factors as a function of such parameters as disk flexibility, blade density, and the total number of assembled groups. It is also shown that many characteristics of the system spectra of natural frequencies are strongly dependent on the number of blade groups.

INTRODUCTION

The techniques of modal analysis and the study of the modal characteristics of blades and bladed-disk assemblies have received significant attention in the literature (e.g., see Platzer and Carta, 1988). Few investigations, however, have been concerned with the forced response of such structures, and this is particularly evident for the study of grouped, or packeted, blades. Compared to individual blades, the vibration spectra for blade groups is typically composed of many more modes with the potential to respond to harmonically-generated excitation. The situation heightens in complexity with the inclusion of a flexible disk, which requires consideration of the entire system of disk and blades for analysis, and is accompanied by an increase in the number of modes. However, it has been shown (Wagner and Griffin, 1993) that not all of these modes have the potential to respond to harmonic forcing.

One of the earliest notable studies of grouped-blade harmonic response was presented by Weaver and Prohl in 1958, who used a newly developed calculation procedure to categorize mode patterns, single mode resonant response factors, and stresses for tangential, axial, and torsional stimuli due to nozzle wake excitation. Although limited to groups of parallel blades, this work was notable for its early use of the digital computer, and the development of a technique which has been widely used in the industry for many years.

In 1977, Thomas and Belek used a finite element model for free-vibration parameter studies of shrouded blade groups with parallel blades. The effects of various weight, flexural rigidity, and length ratios on the tangentially-directed (i.e., in the plane of the supporting disk) natural frequencies were categorized. They also showed that the in-plane vibration characteristics of a symmetric cross section blade group could be predicted from an inference diagram for a two-blade group. Salama and Petyt (1978) did a similar study, but additionally examined the effects of various positioning arrangements for coupling wires between blades, the effects of finite radius of the disk (i.e., radially oriented blades), and the stiffening effects of rotation. This analysis constrained the geometry, and thus the mode shapes, to be symmetric. Of significant note are the findings...
that grouping tends to cause natural frequencies to occur in families of $M$ frequencies, where $M$ is the number of blades in a group, and each family tends to be closely related to the various modes of an individual blade with tip mass. Since the rigidity of the shroud is usually high relative to the blades, there also results a subdivision of frequencies within a family, causing one isolated natural frequency and a relatively narrow band of $M-1$ frequencies. The isolated mode occurs at the lowest frequency with all blades in-phase, and the other modes have various out-of-phase mode patterns. The same conclusions were drawn by Bernante, Macchi and Magnenachi (1982), without the constraint of symmetric group geometry.

Sabuncu and Thomas (1992) developed a comprehensive finite element, beam-based model for a grouped blade system. The effects of pretwist, stagger angle, rotational speed, shroud length, shroud thickness, shroud width, and the distance of the shear center from the centroid were investigated. One interesting finding is that increasing the shroud mass decreases the fundamental frequency under zero rotation, but as a result of centrifugal stiffening, a heavier shroud tends to raise the fundamental frequency as rotational speed increases. Comparisons between theoretical and experimental results showed good agreement.

One of the first in-depth investigations into the excitation of rotationally-periodic structures, such as blade groups on a flexible disk, was done by Wildheim (1979), who examined the response of such structures to a concentrated, rotating force. He noted that a resonance condition exists in an $n$ nodal diameter mode whenever the natural frequency $\omega_n$ is given by,

$$\omega_n = (kN \pm n)\Omega$$

where $\Omega$ is the angular velocity of the force, $N$ is the number of substructures, and $k = 0,1,2,...$. Wildheim introduced a frequency versus nodal diameter diagram, coined the ZZENF diagram (Zig-Zag Excitation line in Nodal diameters versus Frequency), that indicates potential responses at points of intersection of the natural frequency curves and the excitation lines. The work of Singh and Vargo (1989) made use of a graph similar to the ZZENF diagram, referred to as the SAFE diagram, to elucidate certain grouped-blade experimental results presented by Weaver and Prohl (1958), and others.

In 1980, Ewins published a study of blade groups on a flexible disk, concluding that the grouping of blades leads to a gross form of mistuning, causing many diametral components. This implies that a given harmonic does not excite a pure mode. However, Ewins found that grouping can also cause the response of certain diametral modes to be suppressed for harmonic excitation. This was also the conclusion of Ortolano, LaRosa and Welch (1981), who undertook an industry-oriented study of long blade groups. They showed, both theoretically and by several practical examples, that if a blade group is of the same circumferential length as the period of the excitation, response in the lowest frequency, in-phase mode will be minimized. This form of grouping has been referred to as “harmonic shrouding.”

Several of the previously cited investigators have presented experimental results for individual blade groups on a relatively rigid disk, but results from experimental studies of grouped blade and disk system dynamics have seldom been reported in the literature. One such study by Ewins and Imregun (1984) presented the results of a comparison of experiment with two methods of modal analysis for grouped-blade vibration. A series of experiments using constant thickness beam and plate models were conducted primarily to confirm the numerical predictions. The direct method of analysis, which considered the entire structure without using symmetry arguments, proved to match the experimental results quite well. The second method of analysis used assumed nodal diameters, and although many of the results agreed with the measurements, additional modes were predicted (i.e., “ghost” modes) that could not be found experimentally. A method was presented for predicting the nodal diameter components by the construction of a modal interference diagram using the basic geometry of the blade assembly.

In 1985, Pfeiffer experimentally investigated the vibratory behavior of coupled steam turbine blades. Resonant stresses were measured for free-standing and coupled blades using strain gages mounted on the blades. Vibratory response patterns were illustrated using holographic techniques. It was observed that grouping caused each of the group modes to be constrained to a limited range of nodal diameters with little change in natural frequency, and as the stage responded to higher harmonic excitations, each of the frequency ranges would “step” to the next range. At each of the steps, there were also two natural frequencies corresponding to the same nodal diameter. The lower frequency could be viewed, for example, as a mode with nodes between groups, while the higher frequency exists with nodes within the groups. Similar experimental observations were reported by Ewins and Imregun (1984) using simple beam and plate models.
It is fair to say that grouped-blade systems, being structures that possess rotational periodicity, require significantly more complex analysis than individual cantilevered blades. Although much research has been devoted to faster and more comprehensive numerical techniques, much less effort has been directed toward studies of the vibratory behavior of a cyclic system. One example of such an investigation was recently undertaken by Wagner and Griffin (1993) using a continuous string model to represent the motion of a system of grouped blades on a flexible disk. This work emphasized that an understanding of the associated vibratory phenomena must be available to turbine design engineers before blade groups can be optimally designed for efficiency and reliability. Accordingly, the objective of the present work is to qualitatively study certain aspects of grouped blade dynamics by varying a number of significant parameters. The “mode participation factor” is used as a relative measure of response to harmonic excitation, and the computation of this factor is based on the methods presented in the Part I companion paper (Wagner and Griffin, 1994). When this factor is combined with damping and the frequency margin from resonance, stresses and displacements can be computed for each mode. Since the participation factor is loosely defined as the inner product of the excitation distribution and mode shape, it is clear that modal analysis must be the initial priority.

In the present paper, the theory of cyclically-symmetric structures is applied to a representative problem of grouped blades on a flexible disk. After defining the geometry and material characteristics for the model problem, and comparing certain results to those of a continuous analog model for grouped blades, results for both modal and response analyses are presented. The response analyses are based on the use of participation factors. Mathematically, the amplitude of the resonant response is equal to the participation factor multiplied by the dynamic magnification factor for the mode of interest. Consequently, the participation factors provide a convenient method for assessing the relative level of vibration that would result from design changes. Furthermore, they would provide a reasonably accurate quantitative assessment of the resonant response if the magnitude of the excitation
\[ h = jN \pm m \]
where \( j = 0, 1, 2, \ldots \) and the further condition that \( h \) cannot be negative. It was shown in Part I that the condition required for an \( N \) group bladed disk to respond to an excitation harmonic \( h \) while constrained to mode harmonic \( m \) is given by \( h = jN \pm m \) and the modal damping were accurately known.

1 In this study it is assumed that the magnitude of the harmonic excitation is constant. This is usually a reasonable assumption provided the rotational speed of the machinery is limited in range.

DETERMINATION OF THE EXCITED MODES

As discussed in the Part I paper (Wagner and Griffin, 1994), the modes of a system of grouped blades can be organized into groupings of modes: each grouping, or “mode harmonic,” related to a particular value of circumferential displacement phase between boundaries of a cyclically symmetric sector of the structure, which contains a single blade group. The modes within a mode harmonic grouping have mode shapes similar to the various bending or torsional modes of a single beam, but generalized to the group motions (e.g., 1st group bending, group torsional or X mode). A “mode family” will be defined, for the purposes of this study, as the set of modes having similar group motions, but with each family member having a different mode harmonic value. Thus, for example, a group bending mode moving in phase with adjacent groups and having all intergroup blades moving in phase, is in the same family as a similar intergroup mode that is out-of-phase with adjacent groups. The mode families can be identified by the order of frequencies, with the first family having the lowest set of frequencies, the second family the next highest, etc.

Typical grouped-blade modal results tend to show natural frequencies that are relatively tightly clustered. However, the modes within these clusters belong to different mode harmonic groupings, and it has been shown (Wagner and Griffin, 1994) that different mode harmonics respond independently to various orders of harmonic excitation. In fact, if a mode responds to a given excitation harmonic, it tends to be orthogonal to the excitation patterns of nearby excitation harmonics. Thus, the individual modes with the potential to respond to given exciting harmonics tend to have well isolated natural frequencies, and it follows that the use of participation factors for harmonic stress and displacement response calculation, which depends upon mode separation, should be very effective.

It was shown in Part I that the condition required for an \( N \) group bladed disk to respond to an excitation harmonic \( h \) while constrained to mode harmonic \( m \) is given by

\[ h = jN \pm m \]

where \( j = 0, 1, 2, \ldots \), with the further condition that \( h \) cannot be negative, and that negative values of \( m \) need not be considered. This condition was expressed in tabular form by Wagner and Griffin (1993) in a mode-excitation orthogonality diagram, which is shown for the first mode family (i.e., all blades within a group vibrating in phase) in Figure 1.
The cells containing an X indicate intersecting values of h and m that combine for non-zero response. All other combinations result in mode and excitation patterns which are orthogonal, leading to zero response in the mode. The diagram shows that although a multiplicity of excitation harmonics can excite a given mode, a particular excitation harmonic must excite a unique mode. Thus, the relationship between the mode harmonics (and therefore the natural frequencies) and the excitation harmonics is periodic. This observation will later be demonstrated graphically based on numerical computations.

Out of the many modes available in the complex, grouped-blade system, the establishment of the only mode (or degenerate mode pair) within a mode family that will respond to a given harmonic input can be found using equation (2) or the diagram of Figure 1.

<table>
<thead>
<tr>
<th>h</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>N/2</th>
<th>...</th>
<th>N-2</th>
<th>N-1</th>
<th>N</th>
<th>N+1</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>X</td>
<td>-</td>
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<td>...</td>
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<tr>
<td>1</td>
<td>X</td>
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<td>X</td>
<td>X</td>
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<td>...</td>
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<tr>
<td>N/2</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>...</td>
<td>X</td>
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</table>

a) Even number of blade groups

<table>
<thead>
<tr>
<th>h</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>N-2</th>
<th>N-1</th>
<th>N</th>
<th>N+1</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>X</td>
<td>-</td>
<td>-</td>
<td>...</td>
<td>-</td>
<td>...</td>
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<td>-</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>X</td>
<td>-</td>
<td>-</td>
<td>...</td>
<td>-</td>
<td>...</td>
<td>X</td>
<td>X</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>X</td>
<td>...</td>
<td>X</td>
<td>...</td>
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<td>X</td>
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<tr>
<td>N-1</td>
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<td>-</td>
<td>X</td>
<td>...</td>
<td>X</td>
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<td>-</td>
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<td>-</td>
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<td>2</td>
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<td>X</td>
<td>...</td>
<td>X</td>
<td>...</td>
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</tr>
</tbody>
</table>

b) Odd number of blade groups

Figure 1: Mode-excitation orthogonality diagram for mode family 1

Determining the level of the response can accomplished by computing the participation factors as described in Part I. These theoretical concepts will be used in the remainder of this paper to parametrically study the grouped-blade model problem.

**DEFINITION OF THE MODEL PROBLEM CHARACTERISTICS**

The bladed-disk system used in the model problem is a simplified, but representative, grouped-bladed disk model for a constant-speed, low-pressure, steam-turbine stage. The "blades" in the model have a rectangular cross-section that is radially unchanged. The total number of blades equals 120, and the number of blades per group will be a variable. Any departures from the geometry of the nominal model will be noted. The shroud is positioned at the blade tips, and has the same cross-section as the blades. The long axes of both blade and shroud cross-sections are parallel to the disk spin axis, and the "root" of each blade is fixed to the disk rim. The disk is of constant thickness and is fixed at the inner radius. The geometric and material characteristics for the nominal blade, shroud, and the disk are listed in Table 1.

<table>
<thead>
<tr>
<th>Table 1: Nominal Discrete Model Characteristics</th>
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</thead>
<tbody>
<tr>
<td>Rotational Speed:</td>
</tr>
<tr>
<td><strong>Blade</strong></td>
</tr>
<tr>
<td>Length:</td>
</tr>
<tr>
<td>Width:</td>
</tr>
<tr>
<td>Thickness:</td>
</tr>
<tr>
<td><strong>Shroud</strong></td>
</tr>
<tr>
<td>Width:</td>
</tr>
<tr>
<td>Thickness:</td>
</tr>
<tr>
<td><strong>Disk</strong></td>
</tr>
<tr>
<td>Inner radius:</td>
</tr>
<tr>
<td>Outer radius:</td>
</tr>
<tr>
<td>Thickness:</td>
</tr>
<tr>
<td><strong>Material Properties (All components)</strong></td>
</tr>
<tr>
<td>Elastic modulus:</td>
</tr>
<tr>
<td>Poisson's ratio:</td>
</tr>
<tr>
<td>Weight density:</td>
</tr>
</tbody>
</table>

Each blade has been discretized into five, equal-length, beam-type finite elements based on Bernoulli-Euler theory. The shroud is represented by two beam elements between each blade. The disk is constructed from eight-noded isoparametric plate elements, with three rows of elements radially, and one element column for each blade. A schematic illustration of the discretized, bladed-disk model is shown in Figure 2. Steady loads on the blades have been ignored. The applied harmonic forces are distributed over the length of each blade, with a total amplitude per blade that equals 100 lbs. This force level is assumed to be the same for each harmonic. The model thus constructed can be shown to provide participation factors that have converged to within one percent accuracy for the first two mode families.

Throughout the analysis of the model problem, only the modes that are out-of-plane of the disk, and
thus involve disk coupling, will be considered. The modes that are in-plane (i.e., tangential modes) will be ignored, but these modes will behave in a similar manner and fall into families defined by the mode harmonics whenever disk flexibility is a factor in their development. The simplified model used precludes disk flexibility effects from influencing the tangential modes. Thus, each tangential mode will behave as on a rigid disk, and neither the natural frequency nor the group mode shape will change with mode harmonic.

COMPARISON TO CONTINUOUS, ANALOG MODEL RESULTS

In the paper by Wagner and Griffin (1993), a continuous string analogy was offered as a simple model for understanding the vibratory behavior of grouped blades. The normalized participation factors, $A_{mn}$, were presented as a function of $h/N$, where the index $n$ refers to the particular mode within the $m$th mode harmonic grouping. The participation factor results, similarly normalized, for the first two modes of a discrete model with eight blades per group on a relatively stiff disk (12 inches thick),\(^2\) are presented with comparison to the analog results in Figures 3 and 4. These results can also be considered as baseline, to which the results of the later parameter studies can be directly compared. The relative differences between the first and second mode family results for the two models can be shown to be attributed to the ability of the discrete model to independently consider more detailed blade characteristics (i.e., moments of inertia, torsional stiffness, etc.). For example, the second mode is sensitive to the blade torsional stiffness, which is nearly irrelevant to the first mode response. Consideration of this type of detail is not possible with the analog model.

\(^2\) It is questionable whether a disk of this thickness with the given radial dimensions could, within the associated theory, lead to accurate results for the disk dynamics alone. However, for the purposes here, the thick disk serves simply to provide a very stiff, effectively rigid, constraint at the blade roots.
Although the absolute magnitudes have not been compared, and cannot be without calibration of the analog model, this comparison indicates that the analog model is adept at identifying the basic vibratory phenomenon, particularly the excitation harmonics that result in maximum and minimum response for each mode. However, the shape of the analog model curves will not change when coupling stiffness (i.e., "disk" flexibility) is changed. It will be seen from the discrete model results that disk flexibility and mass does indeed change the shape of the participation factor curves. The analog model is incapable of accurately simulating a relatively flexible disk because of the simple nature of the analog model coupling, and because "disk" mass has been ignored.

MODAL ANALYSIS OF THE DISCRETE MODEL

Accurate modal analysis is important to turbine design because, firstly, this will permit identification of natural frequencies which, if possible, should be avoided during operation. Secondly, both the natural frequencies and the mode shapes are required for the computation of the participation factors.

The purpose here will be to illustrate some of the general characteristics found in the modes of a grouped-blade system, and in particular, the changes in the system modes as the mode harmonic is varied. The model used is the nominal model with eight groups on the disk (15 blades per group). This model was chosen because of the long group length, which permits better visual resolution of the phase differences between groups.

The first two modes, illustrated in Figures 5 and 6, are for the blade group rigidly supported at the base of the blades. The displaced position of the group is indicated by the solid lines, while the undisplaced position is dashed. The first mode has all blades moving in phase, and the second mode occurs with ends out-of-phase (often called the X mode). These modes, because of the rigid constraints at the bases of the blades, have natural frequencies that are upper bounds on the system frequencies.

The bladed-disk system mode shapes for the first two mode families (m = 0, 2, 4) are shown in Figures 7, 8, and 9. (For further illustration, see Wagner, 1993.) The modes shown, and those for m = 1 and 3, are the only possible modes that can be associated with the rigid-disk in-phase mode, and the rigid-disk X mode.

Since the disk has eight groups, there can be only 5 mode harmonics in each family, numbered 0 through 4. Thus, the in-phase mode family cannot exist with any number of nodal diameters greater than 4, and the X mode family does not exist beyond 8 nodal diameters.

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[3] Hereafter, the first mode will be referred to as the "in-phase mode," and the second mode as the "X mode."
In general, we see that as the mode harmonic increases, the in-phase mode becomes more locally constrained near the sector boundaries, and thus increases in natural frequency. This is in contrast to the X-mode family behavior which is less constrained at the higher mode harmonic (Figure 9, m = 4), with antinodes at the sector boundary positions, and develops greater constraint, and higher frequencies, with decreasing mode harmonic until a node develops at the sector boundaries for the zeroth mode harmonic (Figure 7). This type of alternating constraint behavior persists with the higher mode families, and is closely related to the number of displacement phase changes occurring within an individual group.

The zeroth mode harmonic (m = 0) results are shown in Figure 7. By setting m = 0, each group is required to have identical deflections at the edges of the sector. It is clear that this is so for the first mode family with all blades in phase, but the only way for this to occur for the X mode family is for the sector boundary displacements to be zero. The first mode family with m = 0 also corresponds to the lowest zero nodal diameter mode for the system. However, the X mode with m = 0 corresponds to an eight nodal-diameter mode. Thus, the nodal diameters and the mode harmonics are not, in general, numerically identical. The zero mode harmonic will include modes with 0, N, 2N, ..., nodal diameters, and is the only mode harmonic that will always have clearly delineated nodal diameter modes (i.e., a single nodal diameter Fourier component) regardless of whether the number of groups is odd or even.

A common conception in industry is that as the number of nodal diameters increases, the corresponding natural frequencies will also increase toward an asymptote defined by the rigid-disk, upper-bound frequency. This is a reasonably practical assumption for a large number of individual blades on a flexible disk, but there are problems with this concept when it is applied to grouped-blade systems. The first problem with this idea is that there are many modes that are made up of a combination of nodal diameter components, so that it may be difficult to identify a unique nodal diameter number with any given mode. Secondly, and more
importantly, the natural frequency upper limit for any given mode family is usually much lower than the rigid-disk upper bound. This is apparent from the example problem, in-phase mode, upper-limit natural frequency (m = 4) of 379 Hz indicated in Figure 9, when compared to the rigid-disk frequency of 472 Hz (Figure 5) for the in-phase mode. The upper limit natural frequency for the X-mode family is 480 Hz which occurs at the zeroth mode harmonic (m = 0). The corresponding rigid-disk natural frequency is 500 Hz. It is not possible to find modes that will bridge these “gaps” and tend toward an asymptotic relationship. It is clear that the fewer cyclically-symmetric sectors a “tuned” structure has, the more difficult it becomes to associate the frequency variation within a family with an upper-bound asymptote. However, if a structure is “mistuned,”

A “tuned” structure, in this case, refers to perfect rotational periodicity, with all groups identical. This terminology is often used in the literature, and is distinct from the “tuning” required to relatively position a natural frequency in a desired way.

that is, possessing some imperfection in rotational periodicity, many more modes may become apparent in the frequency spectrum, and in such a case, it may be possible to measure resonances that occur near the upper-bound frequency.

PARAMETER VARIATION USING THE DISCRETE MODEL

As a demonstration of the ability of the discrete model to consider practical design alternatives, the results of several parameter variation studies are presented. These studies will concentrate on participation factor variation as a parameter of interest is changed. Since the participation factor, as defined in this work, is related to the displacement response, no concurrent conclusions can be drawn regarding stress response without additional considerations. With disk flexibility variation, for example, greater disk flexibility generally results in participation factors of greater magnitude in the lower frequency modes, which implies greater blade-tip displacements. However, stresses at the blade root may tend to reduce as disk flexibility is increased, since the relative blade-tip to disk-rim displacements become smaller. Both the disk flexibility and blade density studies in the following two sub-sections are based on perturbations of the nominal geometry.

Effect of Disk Flexibility

As seen from Figures 10 and 11, increasing the disk flexibility causes an increase in the maximum displacement response of the nominal blade group if the normalized excitation harmonic (h/N) takes on values between an integer and an integer plus one-half. However, response tends to decrease for other values of h/N. The normalized harmonic excitation values for zero response do not change position, but in the first mode family, the harmonics associated with the peak displacements shift to lower values of h/N. These same results have been normalized and plotted on a linear scale in Figures 12 and 13, and here the nearly monotonic variations in the participation factors with disk thickness are easily seen.

An interesting variation can be observed in the second mode family participation factor plots for the two-inch thick disk (Figures 11 and 13). A number of “dips” in the curves can be seen at various h/N values. These apparent anomalies are generated by the close proximity of the X mode, 2nd mode harmonic natural frequency, and a nodal circle mode.
natural frequency which is much higher for thicker disks. The mode shapes for these two modes share proportions of the normal characteristics of each, making categorization difficult. The phenomena associated with these effects is described in the theory of frequency curve crossing or veering (Perkins and Mote, 1986).

**Effect of Blade Density**

The harmonic forces applied to the beam-type, blade model have been simply "sampled" from the harmonic distribution. No attempt has been made to circumferentially integrate the forces, and to apply them as a resolved force and moment pair. Thus, under these conditions, changing the number of blades per unit circumferential length causes an effect that is similar to changing the group's solidity.

The plots of Figures 14 and 15 indicate the effects of blade density for 4, 8, and 16 blades per group, while keeping the total number of groups equal to 15. The most notable effect in the first mode family is that low values of blade density result in unusually high participation factors for higher harmonic excitations. This is also apparent for the second mode family, and is a result of "aliasing," which is a term used in the theory of signal processing, and refers to the inability of the groups to discern...
differences between the sampling of certain lower harmonics and related higher harmonics (see Chapter 15 of Platzer and Carta, 1988). Lower blade density also introduces new minima at integral values of h/N in the second mode family.

![Figure 14: Effect of group solidity by changing number of blades per group — mode family 1](image)

The four blades per group results appear to "fold" or "mirror" as a result of aliasing at an h/N value of about 2. Although not shown, numerical results for higher h/N values show that the eight blades per group results fold at h/N = 4.

**Effect of Number of Groups**

Once the blade aerodynamic designers have finalized the details for the blade passages, the mechanical designers must decide on the blade structural details that will provide optimum reliability. One way to accomplish this task, without affecting the shape of the blade passages, is to evaluate alternative grouping arrangements. It may often be the case that one or more dominant harmonic excitations are known to exist, due to some upstream blockage condition, with some form of harmonic shrouding being the best choice for grouping. Otherwise, grouping can be chosen to avoid resonance conditions, and to provide the lowest possible participation factors in modes of interest.

Selection of an appropriate number of groups is a matter of keeping track of the excitation harmonic h, and the associated mode harmonic m, along with the natural frequencies and participation factors. The excitation and mode harmonics are related as indicated either by equation (2) or the mode-excitation orthogonality diagram of Figure 1. Also, as previously indicated, each mode family can have only one mode for each mode harmonic. Thus, as the excitation harmonic number increases, the mode harmonic number cycles periodically between the limits 0 and N/2 (or (N-1)/2 for N odd). This upper limit value of the mode harmonic will be referred to as the "cutoff harmonic." Any excitation harmonic number that is larger than the cutoff value will potentially cause response only in the modes that have already been defined below the cutoff, albeit with a different participation.

A plot which clearly shows the periodic nature of the natural frequencies when plotted versus the excitation harmonic is shown in Figure 16. Curves are displayed for five different group lengths, and the cutoff harmonic values are indicated. The points shown on these curves represent the available modes in the system, and the connecting lines are shown only to emphasize association. Also shown is the excitation line which defines the excitation frequency as a function of the excitation harmonic for the fixed rotational speed of 3600 rpm. Thus, any mode points on the various curves, lying on or near the excitation line, define potential resonance or near-resonance conditions.

Examination of Figure 16 shows that a greater number of groups tends toward a broader range of frequencies which are available for excitation. For the cases shown, a disk having 30 groups has a frequency range of 223 Hz, while a 6 group system has a frequency range of only 64 Hz. These ranges are, of course, a strong function of the disk flexibility.

The choice of an appropriate group length also depends on minimizing the participation factor for the most important modes. Table 2 gives a list of mode
Table 2: Natural Frequencies and Participation Factors for Responding Modes with Different Group Lengths

<table>
<thead>
<tr>
<th>Mode Family 1</th>
<th>Mode Family 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group Length</td>
<td>Frequency (Hz)</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>15</td>
<td>7</td>
</tr>
<tr>
<td>30</td>
<td>8</td>
</tr>
</tbody>
</table>

A more convenient presentation of the information shown in Figure 16, for a given number of groups, can be obtained by "folding" the curves about the cutoff harmonic value. Such a construction is shown in Figure 17 for eight groups, and is referred to as a ZZENF diagram as originally proposed by Wildheim (1979). The folding makes no change in the mode frequency values, due to their periodicity relative to the cutoff harmonic, but the excitation harmonic line becomes a "zig-zag" with various branches; each branch representing a range of excitation harmonic values. For the 8-group case shown, we can see that the first mode family can potentially be excited by a fifth excitation harmonic in the third mode harmonic, and the second mode family is most vulnerable to the eighth excitation harmonic in the zeroth mode harmonic.

Figure 17: ZZENF diagram for the lower two mode families of an eight-group bladed disk

CONCLUSION

A given excitation harmonic can excite only a limited subset of the structural modes of a perfectly periodic, packeted bladed disk. The modes that can be excited are most easily calculated by analyzing a single blade group and disk segment with a specific cyclic symmetric boundary condition, i.e., the only modes that can be excited are those that have a phase difference across the segment that is identical to the phase difference associated with the excitation harmonic.

Once a structural mode is identified as capable of responding to a given harmonic excitation, then the relative strength of its resonant response is given in

...
terms of its participation factor. Its participation factor may be efficiently calculated by using the modal information from a cyclically symmetric, finite element analysis of a single group. This approach is utilized in order to illustrate how various design parameters affect the resonant response of grouped blades.

The structural modes can be classified as belonging to families that exhibit similar behavior. When the participation factors of a family are plotted as a function of the excitation harmonic divided by the number of groups, they show a trend similar to that of the magnitude of a decaying cosine function. In the case of the lowest frequency family, the first zero occurs when the excitation harmonic equals the number of groups. This condition, and the associated grouping configuration called harmonic shrouding, is well-known and frequently exploited to minimize the response of the system’s first mode to a known harmonic excitation. In fact, the participation factors of all modal families exhibit similar trends and the harmonic shroud concept can be readily applied to any mode and excitation harmonic pair once the participation factors for a family are calculated as a function of the normalized excitation harmonic.

In general, however, the designer is often concerned with more than a single mode/excitation harmonic pair, and trade-offs must be made. If a participation factor is not zero, then its value is affected by parameters such as the thickness of the disk and the blade density of the stage. In fact, under these more general circumstances, there is no simple rule for optimally choosing the number of groups. However, an approach has been defined which allows the designer to establish which cases are critical, and then, the participation factors can be calculated for each case to establish the number of groups that will result in the most robust design.

REFERENCES


