SLIDING MODE OUTPUT FEEDBACK CONTROL
OF A FLEXIBLE ROTOR VIA MAGNETIC BEARINGS

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ABSTRACT---A sliding mode feedback algorithm is proposed to control the vibration of a flexible rotor supported by magnetic bearings. It is assumed that the number of states is greater than the number of sensors. A mathematical model of the rotor/magnetic bearing system is presented in terms of partial differential equations. These equations are then discretized into a finite number of ordinary differential equations through Galerkin's method. The sliding mode control law is designed to be robust to rotor unbalance and transient disturbances. A boundary layer is introduced around each sliding hyperplane to eliminate the chattering phenomenon. The results from numerical simulations are presented which not only corroborate the validity of the proposed controller, but also show the effects of various control parameters as a function of the angular speed of the rotor. In addition, results are presented that indicate how the current required by the magnetic bearings is affected by control parameters and the angular speed of the rotor.

INTRODUCTION

Rotord vibration has been an important problem since the development of turbomachinery and electrical machines (Vance, 1989; Dimarogonas and Paipeitis, 1983; and LaLanne and Ferraris, 1990). One of the key sources of vibration is the inevitable rotor unbalance and the resulting dynamic force that is proportional to the square of the rotor speed. Hence, rotor vibration is a more severe problem for machines operating at higher speeds. In fact, for good performance at higher rotational speeds, it is necessary to have smaller amplitudes of vibration than for the same performance at lower speeds (Rao, 1995). One approach that has been taken to reduce the severity of rotor unbalance is rotor balancing, a technique that essentially adds a mass to the shaft that leads to wear problems. Further, it has been found that it is difficult to accurately calculate the point of force application (Bobbert and Schamhardt, 1990). Conversely, magnetic bearings require no lubrication and do not contact the shaft thus avoiding wear problems. Hence, a significant amount of research has been devoted to develop a highly reliable magnetic bearing system for active vibration control.

It is well known that the attractive force of each magnetic bearing varies proportionally to the square of the coil current produced by the magnetic bearings and inversely proportional to the square of the air gap between the rotor and stator. As a result, when the air gap decreases, the attractive force increases for a fixed coil current and the object is brought closer and closer to the magnet. Conversely, when the air gap increases, the object is continuously drawn away from the magnet. This phenomenon is known as negative stiffness and results in an inherently unstable system. Thus, for a stable configuration, active control must be employed. An active suspension system continuously monitors the object's position and applies a force to return the system to a stable configuration. Some details of magnetic bearing design, in particular the integration of displacement and velocity sensors within the magnetic bearing, were discussed by Ulbrich and Anton (1984). Sinha (1990) has summarized the basic issues associated with the stability of active magnetic suspension systems. In addition, Humphries et al (1986) reported the effects of control algorithms on magnetic bearing stability. More specifically, they studied differences in electronic circuit bandwidths, sensor positioning, and the overall bearing stiffness and damping coefficients. Allaire et al (1986) investigated critical speeds and unbalance response of a flexible rotor in magnetic bearings. Modal control of a flexible rotor was demonstrated by Salm and Schweitzer (1984).

One of the most important problems involved in the control of rotors is the rotor unbalance caused by the unavoidable uncertainties in the rotor eccentricity. Consequently, algorithms that can be designed to compensate for these uncertainties are advantageous. Sliding mode control theory (Utkin, 1977, 1983) is one such theory that lends itself to the design of robust controllers. The power of this technique is that a controller can be designed for robustness with respect to uncertainties and transient disturbances provided their bounds are known. This is in contrast, for example, to a PID controller whose gains must be re-tuned when changes in unbalance occur. Although quantifying the exact amount and location of the unbalance is quite difficult, it is generally a straightforward process to establish an upper bound of the unbalance force.

Sinha et al (1991) have considered sliding mode control of a rigid rotor supported by magnetic bearings. However, many rotor systems...
cannot be treated as rigid and flexible effects must be taken into account. Rundell et al (1996) used a sliding mode observer and controller to stabilize the rotational motion of a vertical shaft magnetic bearing. To develop their controller, they needed estimates of the derivatives of the disturbance. They also used a matrix to help account for robustness and gave an example of a matrix computed experimentally, but they did not present a procedure for determining the matrix on a general basis. Tian and Nonami (1996) applied discrete-time sliding mode control to a flexible rotor/magnetic bearing system. They presented the design of a robust observer requiring several matrices to exist, but they did not give methods for finding these matrices. Nonami and Ito (1996) investigated μ-synthesis of a flexible rotor/magnetic bearing system. However, this approach involves a complicated iterative and nonconvex numerical procedure. This paper explores a new approach to the robust control of a flexible rotor, using only output feedback.

Using the upper bound of the unbalance, a controller is to be developed that will guarantee robustness as long as the eccentricity is under this bound. It is a straightforward process to guarantee robustness with sliding mode control if all the states are known which is a good assumption for a rigid rotor. In the context of active control of flexible rotors via magnetic bearings, the number of effective actuators and sensors is generally small not only when compared to the dimension of the full-order model, but also to the number of the states of the reduced-order model. As a result, not all the states are available for measurement. It is found that the design of the sliding mode controller that must be developed on the basis of outputs requires the upper bounds of the states. This is a coupled problem since the states are influenced by the controller and the bounds on the states are required to design the controller. Yallapragada et al (1996) and Wang and Fan (1994) have examined sliding mode controllers based on output feedback. Recently, Lewis and Sinha (1997,1995) have addressed this problem with a novel sliding mode control output feedback scheme for a mechanical system.

This paper presents an application of the control methodology formulated by Lewis and Sinha (1997,1995) for the control of a flexible rotor via magnetic bearings. The model of the rotor system is developed through Hamilton’s principle and a technique is given to reduce the order of the model for control law development. Numerical computations, which determine the gains necessary to ensure robustness to uncertainties in the rotor eccentricity, are presented to corroborate the applicability of the approach to the rotor problem. Also, results are presented to show the effects of control parameters as they relate to steady-state amplitude, current requirements of the magnetic bearing forces at the left end of the shaft in the x and y directions, respectively. Similarly, \( F_{bx} \) and \( F_{by} \) are the magnetic bearing forces at the right end of the shaft. Relationships between these forces and currents in the magnetic bearing actuators can be found in the Appendix. The \( p_i \) and \( q_i \) are generalized coordinates and \( N \) is the number of terms in the Galerkin expansion. The rotor unbalance is \( p \sigma e^2 \) where \( p \) is the mass of the unbalance, \( e \) is the eccentricity, and \( \Omega \) is the angular velocity of the shaft.

In this paper, control algorithms are developed based on output feedback. From Fig. 1, the sensors are located at two axial locations along the shaft, namely at \( z = a \) and \( z = b \), providing both position and velocity feedback. Based on this, two output vectors, one for position and one for velocity, can be formed as

\[
Y_1 = [y_1, y_2, y_3, y_4]^T = Cr \\
Y_2 = [\dot{y}_1, \dot{y}_2, \dot{y}_3, \dot{y}_4]^T = C \ddot{r}
\]

where \( y_1 \) and \( y_2 \) are the displacements in the x and y directions, respectively, on the left end of the shaft, \( y_3 \) and \( y_4 \) are displacements in the x and y directions, respectively, on the right end of the shaft, and \( C = B^T \) is a 4 x 2N matrix.

Reduced-Order Model

Since it is not practical to implement a large order controller, it is necessary to reduce the order of the model to that of a manageable size. In the normal modes approach, the eigenvectors of \( \bar{M}^{-1} \bar{K} \), the undamped system without gyroscopic effects, are considered in the following expansion:

\[
r = \Phi \zeta = \sum_{j=1}^{n \times N} \Phi_j \zeta_j
\]

where \( \Phi \) is the \( N \times n \) matrix of the eigenvectors \( \Phi_j \) of the modes to be retained, \( \zeta \) is the \( n \times 1 \) vector containing the reduced-order coordinates.
The eigenvectors \( \Phi \) have been mass normalized such that \( \Phi^T \Phi = I \). Substituting Eq. (7) into Eq. (1) and premultiplying by \( \Phi^T \) yields the reduced-order system:

\[
\Phi^T X = \Phi^T (\Phi^T C + \Phi^T D) \Phi \zeta + \Phi^T F_c + \Phi^T D \bar{F}_d(t)
\]

Letting \( \Phi^T (\Phi^T C + \Phi^T D) \Phi = K_G \) and \( K_G \Phi \zeta = \Phi^T \Phi \zeta \), and \( X_1 = \zeta \), and \( X_2 = \dot{X}_1 = \zeta \). Then Eq. (8) can be written in first order form as

\[
\begin{bmatrix} X_1^T \ X_2^T \end{bmatrix} = \dot{X} = AX + BF_c + D_d
\]

where

\[
A = \begin{bmatrix} 0 & I \\ -K_s & -D_s \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \Phi^T \Phi \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ \Phi^T \Phi \end{bmatrix}
\]

Finally, using Eqs. (5) and (6), the output vector \( Y \) is written as

\[
Y = C \Phi \Phi^T X = C^* X
\]

### DEVELOPING THE SLIDING MODE CONTROL SYSTEM

Since there are four inputs, four sliding hyperplanes are defined (Asada and Slotine, 1986) as

\[
s_i = (\phi_i + \lambda_i)^2 \int_0^T y_i(t) \, dt \quad i = 1, 2, 3, 4
\]

where the \( \lambda_i \) (i = 1, 2, 3, 4) are control design parameters and \( y_i \) is the displacement at the \( i \)th output.

With a properly designed sliding mode control system, the system states reach a desired manifold (intersection of sliding hyperplanes) and stay on the manifold thereafter. These two stages are referred to as the reaching and sliding conditions, respectively. Following the approach by Lewis and Sinha (1997, 1995), the vector of control forces, \( U \), that satisfy both the reaching and sliding conditions is expressed as

\[
U = -P^* C^* \dot{X} - P^* Y - K_G \bar{F}
\]

where

\[
K_G = \text{diag}(k_1, k_2, k_3, k_4)
\]

\[
\bar{F} = \begin{bmatrix} P_{L1} \\ P_{L2} \\ P_{L3} \\ P_{L4} \end{bmatrix}
\]

and the \( \phi_i (i = 1, 2, 3, 4) \) are the boundary layer thicknesses used in the saturation function (sat) to eliminate the chattering phenomenon. They are computed (Asada and Slotine, 1986) as \( \phi_i = k_i / \lambda_i \). \( Y \) is the output and \( P^* \) is a function of the design parameters. The effect of these parameters is investigated later in the paper. Proper selection of the diagonal gain matrix \( K_G \) ensures that the reaching condition \( (s_i s_j < 0) \) is satisfied. It is a function of the system states, but not all of the states are available for measurement. This problem is circumvented through proper selection of the estimated state vector, \( \hat{X} \).

To illustrate this point, the closed-loop control system is obtained from Eq. (9), noting that \( F_c = (\bar{P} C^* B)^{-1} U \) and using Eq. (13), as

\[
\dot{X} = A \hat{X} + B \bar{P} C^* A \hat{X} + D \bar{F} + \bar{F} \bar{K}_G \bar{F}
\]

The estimate of the state vector, \( \hat{X} \), is defined as follows:

\[
\dot{\hat{X}} = (\bar{A} + B \bar{P} C^* A) \hat{X} + \bar{B} \bar{K}_G \bar{F} \quad ; \quad \hat{X}(0) = \hat{X}_0
\]

where

\[
\bar{A} = A - B (\bar{P} C^* B)^{-1} P^* C^* \quad \bar{B} = -B (\bar{P} C^* B)^{-1}
\]

The choice of \( \bar{X} \) dynamics (Eq. 21)) is key to the off-line computation of the control gain vector such that the reaching conditions \( (s_i s_j < 0) \) are satisfied. From Lewis and Sinha (1997, 1995), the gain vector \( K_s \), which is the diagonal of \( K_G \) consisting of \( (k_1, k_2, k_3, k_4) \), is

\[
K_s \geq [\bar{P} C^* A] X - \bar{X} + [\bar{P} C^* D] \bar{F}_d + \eta
\]

which requires the upper bounds of the errors in the state estimate \( (X - \bar{X}) \) and the dynamic force created by the rotor unbalance. The vector \( \eta \) consists of small positive constants \( \eta_i (i = 1, 2, 3, 4) \). The upper bound on the magnitude of \( (X - \bar{X}) \) can now be obtained from Eqs. (20) and (21) as

\[
|X - \bar{X}| \leq C \int_{-\infty}^{0} e^{\lambda (t - \tau)} D \bar{F}_d(t) \, dt
\]

which is independent of \( K_G \). Thus for a given bound on the initial conditions of \( X \), the magnitude of \( (X - \bar{X}) \) can be determined for any time \( t \) provided the bound of the unbalance force, which is directly related to the upper bound on the eccentricity, is known. Hence, though the unbalance force and initial conditions may not be known exactly, the reaching condition can still be satisfied as long as the bounds of these quantities are known, which is a reasonable assumption.

### COMPUTER SIMULATIONS AND NUMERICAL RESULTS

The primary focus of this paper is controlling the first flexible mode of the rotor/magnetic bearing system in the presence of uncertainties in the rotor unbalance. More modes could be taken since the algorithm with output feedback is still applicable except that the order of the system matrix will increase. For simplicity, only one mode is used here to illustrate the theory. The system can be made robust to the uncertainty in the rotor unbalance provided its bound is known.

Table 1 displays the rotor and magnet parameters used for all numerical calculations. To determine the first flexible mode of the rotor/magnetic bearing system, the full-order model given by Eq. (1) is used. In determining the first flexible mode, it is necessary to determine how many terms are needed in the Galerkin expansion to achieve an acceptable accuracy level. It is determined that \( N = 10 \) provides sufficient accuracy which leads to a full-order state space model with \( 40 \) states. The first open-loop critical speed of the rotor based on this model is 6963 rpm. With the retention of only the first flexible mode and the rigid body modes, the order of the A matrix in Eq. (9) is \( 12 \times 12 \). Subsequent analyses are based on this reduced-order model.
where and thus can be described by the following set of linear equations:

that the system remains inside the boundary layer after 0.01 seconds
speed of 6963 rpm resulting from the integration of Eqs. (20) and (21)
as the control parameter, A, on the response of the closed-loop system.
established, it is desired to ascertain
Steady-State Response
Table 2: Analytical and Simulated Upper Bounds
analytical upper bounds are greater than or equal to all the simulated
values. The gain vector, \( \mathbf{K}_G \), is calculated off-line and the key to insuring robustness to rotor unbalance and transient disturbance, is

Verification of Algorithm to Compute Control Gains
To prove the validity of Eqs. (23) and (24), numerical simulation results are presented for the critical speed of 6963 rpm (116 Hz). The constants \( \eta \) and \( \lambda_i \) are taken to be 0.5 N/kg and 1000 rad/s, respectively, for each of the four sliding hyperplanes. The bound of the unbalance is 9.52 Newtons and only state 1 is assumed to have a non-zero initial condition upper bound. With this bound of 2.01e-04 meters, the upper bounds on \( (X - \hat{X}) \) are determined from Eq. (24) and the gains from Eq. (23). These gains are used in the numerical simulation of the state equations. From the simulation, the largest value of \( (x - \hat{x}) \) is obtained and this value is compared to the value obtained analytically from Eq. (24). Inspection of Table 2 shows that the analytical upper bounds are greater than or equal to all the simulated values. The gain vector, \( \mathbf{K}_G \), calculated off-line and the key to insuring robustness to rotor unbalance and transient disturbance, is

Table 1: Rotor and Magnet Parameters

Table 2: Analytical and Simulated Upper Bounds

Steady-State Response
With a novel approach for determining the control gains established, it is desired to ascertain the effects of these gains as well as the control parameter, \( \lambda_i \), on the response of the closed-loop system. Figure 2 shows a plot of \( s_1(t) \) versus time of the rotor for an operating speed of 6963 rpm resulting from the integration of Eqs. (20) and (21) where \( \phi \) is the boundary layer thickness. Inspection of Fig. 2 indicates that the system remains inside the boundary layer after 0.01 seconds and thus can be described by the following set of linear equations:

where

The steady-state amplitude of \( y(t) \) under the conditions above is 5.77e-06 meters. At the center of the shaft, where the rotor is located, the steady-state amplitude is 9.55e-05 meters. With \( v = [x \ x' \ s]^T \), Eq. (25) can be expressed as

where \( \mathbf{A}_{xy} \) and \( \mathbf{D}_{yy} \) are appropriately defined matrices. The system is stable if the eigenvalues of \( \mathbf{A}_{xy} \) lie in the left half of the complex plane. Stability issues in the sense of Lyapunov are discussed by Lewis and Sinha (1997). It is shown that the eigenvalues of \( \mathbf{A}_{xy} \) are the eigenvalues of \( \hat{\mathbf{A}} + \hat{\mathbf{B}} \mathbf{F}^T \), where \( \lambda_i \) is guaranteed to be stable since it is always a diagonal matrix with positive elements. For colocated sensors and actuators, \( \mathbf{A}_i \) is stable.

The remaining matrix, \( \hat{\mathbf{A}} + \hat{\mathbf{B}} \mathbf{F}^T \), is stable if the symmetric parts of two symmetrizable matrices are positive definite.

It is desired to study the system under steady-state conditions thus avoiding time-consuming simulations. Since the forcing function \( F_d(t) \) is sinusoidal in nature, a steady-state solution to Eq. (27) can be assumed as

where \( \lambda \) are vectors of unknown elements. These vectors can be determined by substituting Eq. (28) into Eq. (27) and equating coefficients of \( \sin \Omega t \) and \( \cos \Omega t \). The first 12 elements of \( V \) contain the state vector \( \mathbf{X} \); hence, the magnitude of the \( i \)th steady-state output \( Y_i \) is

Results of Parameter Study
In the rotor/magnetic bearing system under study in this paper, there are four magnetic bearing actuators. It is assumed that each of these actuators has the same characteristics and that the bias currents (Appendix) \( I_p \) and \( I_b \) are the same for each magnet pair. It is assumed that there are no parametric uncertainties associated with the magnet parameters. It is also assumed that the value of \( \lambda_i \) is the same for each of the four \( s_1 \) and that the initial conditions are the same for every analysis. Both \( \lambda_i \) and \( k_i \) influence the overall stiffness and damping of the system and thus the frequency content. As with any rotating machinery, the steady-state displacement is a chief concern. And in the case of magnetic bearings, the current necessary to drive the actuator to achieve a desired displacement level is also a primary focus. Since each bearing has the same characteristics and the values of each of the four \( \lambda_i \) is the same, in the analyses that follow, the relationships that exist among the steady-state displacement \( y_1(t) \), \( \lambda_i \), \( k_i \), and \( \Omega_1 \), and incremental current will be explored for one actuator pair. The control law as expressed in Eq.(13) is dependent on two main parameters: the 4x1 vector of design variables \( \lambda_i \), which can be chosen arbitrarily, and the 4x1 gain vector \( \mathbf{K}_G \), that insures robustness to transient disturbances and external excitation. Inspection of Eqs.
SUMMARY AND CONCLUSIONS

This paper deals with active vibration control of a flexible rotor via magnetic bearings using only output feedback. A novel sliding mode algorithm to control the rotor robustly in the event of transient disturbances and external gain excitation has been presented. It is shown that the required gain vector can be calculated easily off-line.

Numerical results demonstrate the validity of this novel control algorithm. Results from parametric studies are also presented. In particular, control parameters required for "good" performance are presented as a function of the angular speed of the rotor. Only parametric uncertainty in the rotor unbalance has been considered in this paper. Future work will be aimed towards parametric error in the rotor model and magnetic bearings.

ACKNOWLEDGEMENT

This work was supported in part by grant NAGW-1356 from the Propulsion Engineering Research Center at Penn State.

REFERENCES


APPENDIX A (Computation of Coil Currents)

It is necessary to relate the control force Eq. (13) to the desired current to be delivered by the magnetic bearings. Figure 1 shows a schematic of the important parameters involved in the calculation of the coil currents. \(F_{ax}\), the force on the left end of the shaft in the x-direction, can be written

\[ F_{ax} = H \left( \frac{l_1^2}{h_y} + \frac{l_2^2}{h_y} \right) - H \left( \frac{l_1^2}{h_0} + \frac{l_2^2}{h_0} \right) h_y g \sin \theta \]  \hspace{1cm} (A.1)

where \(h_y\) is the nominal airgap at equilibrium, \(m_y\) is the total mass of the rotor system, \(g\) is the acceleration of gravity, and \(H = \mu_0 A_p N_p^2\) is a constant associated with the magnets. Here, \(N_p, A_p, \mu_0\) are the number of coil turns, the face area per single pole of magnet, and the permeability of free space, respectively. Similar equations can be established for \(F_{by}\) and \(F_{az}\). In Eq. (A.1), the first and second terms on the right hand side represent the attractive forces of the top and bottom magnets, respectively. The last term accounts for the weight of the rotor system. \(I_{b2}\) and \(I_{b1}\) represent the bias currents in the top and bottom magnets under static equilibrium conditions. The incremental currents of the top and bottom magnets can be either positive or negative and are denoted \(i_{b1}\) and \(i_{b2}\), respectively. The axes of the bearing actuators are located at an angle \(\theta\) from the vertical. The bias current \(I_{b2}\) is calculated as

\[ I_{b2} = \frac{k}{2H} \left( \frac{H h_y^2}{h_0^2} + \frac{l_1}{T + l_1} \right) m_y g \sin \theta \]  \hspace{1cm} (A.2)

Using only the first order terms of the Taylor series expansion and linearizing the control force about the bias currents \(I_{b1}\) and \(I_{b2}\) and the nominal airgap, \(h_0\), the linearized equation, with \(\dot{i}_{b1} = -\dot{i}_{b1}\), is

\[ F_{ax} = \left( \frac{2H l_1^2 + 2H l_2^2}{h_y^2} \right) \dot{i}_{b1} + \left( \frac{2H l_1^2 + 2H l_2^2}{h_0^2} \right) \dot{i}_{b2} \]  \hspace{1cm} (A.3)

The incremental current in the top magnets \(\dot{i}_{b1}\) is solved from Eq. (A.3) and the total coil current for the pair of magnets is

\[ I_{b1} = I_{b2} + \dot{i}_{b1} \]  \hspace{1cm} (A.4)

\[ I_{b2} = I_{b0} + \dot{i}_{b2} \]  \hspace{1cm} (A.5)

Alternatively, since the incremental current and force are linearly related, the original open-loop equations (Eq. 9) could have been modified such that the control law would give the incremental current directly.

Figure 1: Rotor model and bias currents

Figure 2: \(s_1(t)\) and \(\phi\) vs. time for \(\Omega = 6963 \text{ rpm}\)
Figure 3: $k_1$ vs. $\Omega$ for various values of $\lambda$

Figure 4: $y_1(t)$ vs. $\Omega$ at steady state

Figure 5: $y_1(t)$ vs. $\lambda$ at steady state

Figure 6: $\lambda$ for good performance vs. $\Omega$

Figure 7: $i_1$ vs. $\Omega$ for $\lambda$ for good performance