SPANWISE TRANSPORT IN AXIAL-FLOW TURBINES: PART 2 — THROUGHFLOW CALCULATIONS INCLUDING SPANWISE TRANSPORT

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ABSTRACT

In Part 1 of this paper, a repeating stage condition was shown to occur in two low aspect ratio turbines, after typically two stages. Both turbulent diffusion and convective mechanisms were responsible for spanwise transport. In this part, two scaling expressions are determined that account for the influence of these mechanisms in effecting spanwise transport. These are incorporated into a throughflow model using a diffusive term. The inclusion of spanwise transport allows the use of more realistic loss distributions by the designer as input to the throughflow model and therefore focuses attention on areas where losses are generated. In addition, modelling of spanwise transport is shown to be crucial in predicting the attenuation of a temperature profile through a turbine.

INTRODUCTION

Part 1 of this paper described an experimental investigation into the flowfields found in two low aspect ratio, low speed multistage turbines. It was observed that the time-mean flowfield adjusts through the machine as the spanwise gradients of entropy and total pressure develop until a repeating stage condition is reached. The distance over which this development takes place is typically that of two stages. As a significant proportion of the loss within a low aspect ratio turbine is generated on the endwalls, the existence of a repeating stage suggests that the rate of generation of endwall loss, in a global sense, balances the flux of loss away from the endwall regions. Spanwise transport is therefore significant and important. The tracer results presented in Part 1 indicate that in a multistage environment, both turbulent diffusion and classical secondary flow are responsible for spanwise redistribution. In the modelling of spanwise transport it is therefore necessary to consider both mechanisms.

The most common approach for designing turbomachinery is to approximate the flowfield as a series of axisymmetric concentric streamtubes. Until fairly recently there was little recognition in the streamline curvature schemes that the flow was three dimensional and highly turbulent. Adkins and Smith (1982) presented a method that assumed spanwise transport between streamtubes was due to deterministic secondary flows. Gallimore and Cumpsty (1986) argued that turbulent diffusion was the dominant spanwise transport mechanism with Gallimore (1986) presenting a streamline curvature model based on this premise. Wisler, Bauer and Okishi (1987) and Leylek and Wisler (1991) demonstrated that both mechanisms were present with turbulent diffusion being dominant in the mid-span region. Towards the endwall, secondary flow effects were of comparable magnitude.

Subsequent to this debate, a number of papers have been dedicated to throughflow modelling that introduced either or both spanwise transport mechanisms (reviewed by Wennström, 1991). Most investigations to date have concentrated on the application of spanwise transport models to the prediction of multistage compressor flowfields: little attention has been focused on the modelling of spanwise transport in turbines and its significance to flowfield prediction.

Assuming these models are still valid, any of the throughflow models which include spanwise mixing could be applied directly to the prediction of the turbine flowfield. Unfortunately most approaches depend on some semi-empiricism and the models have been tuned for compressor flowfield prediction. In this paper two expressions are presented that enable the scale of spanwise transport due to the two
mechanisms of turbulent diffusion and classical secondary flow to be estimated. A simple model of spanwise transport has been implemented in the updated streamline curvature scheme of Denton (1978) and this scheme is used to predict the flowfields of the turbines presented in Part I and three other applications. The dominant effect of spanwise transport on the radial variation of efficiency and total temperature is demonstrated.

**MODELLING OF SPANWISE REDISTRIBUTION IN STREAMLINE CURVATURE CALCULATIONS**

From a review of the published literature, throughflow models developed for turbine geometries are not as capable of predicting main flow features as those developed for compressors. To the author's knowledge there are few, if any, schemes which can predict the secondary flows in a multistage turbine beyond the use of empirical corrections. This reflects the difficulty in applying inviscid vortex theory to a flowfield that in general experiences high turning, distortion of the Bernoulli surfaces and very thin boundary layers. It also reflects the difficulty in applying boundary layer theory to predict endwall boundary layers that are highly skewed. Methods are available which can predict the flowfield through a single blade row with boundary layers that are a priori defined (Hawthorne and Novak, 1969; Glynn, Spurr and Marsh, 1977; Glynn, 1982; and Gregory-Smith and Ok, 1991). These are claimed to be fairly successful for a single blade row but Hunter (1979) showed that if the method of Hawthorne and Novak (1969) was extended to even a single stage, the predictions became less accurate. This is probably true of all schemes.

The approach taken in this study is that of simplicity at the expense of sophistication and rigour. A model is developed that allows the scale of spanwise transport to be easily estimated, and incorporated within an existing throughflow scheme. This enables a clear picture to emerge in terms of the influence of spanwise transport in turbines, from which further throughflow model development can follow.

The experimental results in Part I showed that spanwise transport takes place due to both turbulent diffusion and convective mechanisms. Spanwise transport however, can only be included within a throughflow model by the introduction of a diffusive term; no mass by definition, can be transferred across a streamtube boundary. The spanwise transport of variable $Q$ is modelled by

$$
\frac{\partial Q}{\partial m} = \varepsilon \frac{\partial^2 Q}{\partial y^2}
$$

(1)

where $m$ refers to the meridional direction. Equation (1) is based on a simplified transport formulation and has a similar form to the model used by Adkins and Smith (1982). The diffusion coefficient $\varepsilon$ is an effective diffusion coefficient determined by contributions from both turbulent diffusion and spanwise convection. In the two low aspect ratio turbines described in Part I, secondary flows were seen to be dominant in effecting spanwise convection and the effective diffusion coefficient is modelled as

$$
\varepsilon = \varepsilon_d + \varepsilon_{sf}
$$

(2)

where $\varepsilon_{sf}$ is based on secondary flow considerations. A conceptual objection to equations (1) and (2) may well arise as the equations implicitly assume that secondary flows, which are inviscid, can be modelled by a diffusive term. Any spanwise convection will however, have a redistributing effect on the spanwise distribution of property $Q$ and, based on physical considerations, be of a diffusive nature to the first order. A more rigorous approach would be to model secondary flow as a convective mechanism by the consideration of the $S3$ transverse plane (De Ruyck, Hirsch and Segaert, 1988). This method which is based on a quasi three-dimensional approach, becomes very complex and ultimately improved predictive capability is not guaranteed. In terms of simplicity and ease of implementation, the model represented by equations (1) and (2) is preferred. A choice is now faced in terms of how $\varepsilon_d$ and $\varepsilon_{sf}$ should be determined and distributed across the span.

**Scaling Model for $\varepsilon_d$**

A similar approach to that of Gallimore and Cumpsty (1986) is taken in determining an appropriate value for $\varepsilon_d$. The production of turbulence is directly related to the generation of loss and therefore entropy. If a repeating stage is considered within a multistage turbine, the turbulence level at inlet to the stage is the same as at exit. Loss is still generated within the stage and therefore the rate of turbulence production must be balanced by the rate of turbulence decay. Although the turbulence may not be initially isotropic, it is presumed that in the global sense the turbulence can be treated as homogeneous and isotropic. This has been shown in Part I to be an acceptable approximation. Within the repeating stage each streamtube experiences the same increase in entropy and change in enthalpy. This does not mean that the rate of generation of entropy in each streamtube is necessarily the same. Assuming two-dimensional, incompressible flow and applying the Second Law of Thermodynamics

$$
T_3 \Delta s = \Delta h_0 - \Delta p_0 / \rho
$$

the definition of isentropic efficiency

$$
\eta = \Delta h_0 / \Delta h_{os} = \Delta h_0 / (\Delta h_0 - T_3 \Delta s)
$$

and the Euler work equation across the rotor

$$
-\Delta h_0 = UV_2 (\tan \alpha_3 - \tan \alpha_2) = UV_2 (\tan \alpha_1 - \tan \alpha_2)
$$
gives the entropy increase in each streamtube

$$
T_3 \Delta s = UV_2 (\tan \alpha_2 - \tan \alpha_1)(1 - \eta) / \eta
$$

(3)

Equation (3) is representative of the entropy production over the stage if the stagnation temperature is constant over the whole span at inlet. The rate of entropy production can be considered to be proportional to the production rate of turbulent kinetic energy. Due to the repeating stage condition it must also be proportional to the dissipation rate of turbulent kinetic energy. The dissipation rate of turbulent kinetic energy per unit volume over the whole stage is therefore given by

$$
\phi = A_d / \eta \Delta s / \rho \Delta V
$$

where $A_d$ is the proportionality constant, $\Delta V$ the volume of the stage and $\rho \Delta s$ the mass flow rate.

Assuming an eddy viscosity formulation and a typical length scale $L_d$ of the turbulent eddies, a form of turbulent viscosity can be determined

$$
\nu_d = (A_d \phi)^{1/3} L_d^{4/3}
$$

$$
\nu_d = \left( \frac{A_d L_d}{\eta} \right)^{1/3} \left( \frac{U (\tan \alpha_2 - \tan \alpha_1)(1 - \eta)}{\nu_2 L_s} \right)^{4/3}
$$

(4)

where $L_s$ is the stage length. Assuming a Schmidt number of unity and non-dimensionalising, an expression for $\varepsilon_d$ is obtained

$$
\frac{\varepsilon_d}{\nu_2 L_s} = \frac{A_d}{\nu_2 L_s} \left( \frac{U (\tan \alpha_2 - \tan \alpha_1)(1 - \eta)}{\nu_2 L_s} \right)^{1/3} \left( \frac{L_d}{L_s} \right)^{4/3}
$$

where $A_d$ is a modified constant of proportionality. Besides $A_d$, the
one variable that is not known in the design phase is the length scale \( L_d \). It can be seen from equation (4) that it has a significant influence on the predicted level of \( \varepsilon_d \) and must therefore be approximated correctly. \( L_d \) has the same order of magnitude as \( \delta \), the boundary layer thickness which is dependent on several parameters including blade loading. An estimate of the momentum thickness \( \theta^* \) can be determined from the profile loss coefficient \( Y_p \), blade pitch \( p \) and blade exit flow angle \( \alpha_2 \).

\[ \theta^* = \frac{Y_p p \cos \alpha_2}{2} \]

Assuming a turbulent boundary layer and a \( (\frac{1}{2})th \) power law the boundary layer thickness \( \delta \) is related to \( \theta^* \) by \( \delta = (72/7) \theta^* \). The length scale \( L_d \) in equation (4) is estimated by

\[ L_d = 5 Y_p p \cos \alpha_2 \]

To establish an appropriate value of \( \overline{X} d \) the experimental data presented in Figures 13 and 14 of Part 1 were used. The mid-span design values of \( \alpha_1, \alpha_2, Y_p \) and the measured overall efficiencies were used in equation (4) to estimate \( \varepsilon_d \). The efficiency of the repeating stage should, strictly speaking, be used based on experimental measurements the differences were not large (Lewis, 1993). A comparison of the experimental data and estimated values obtained by setting \( \overline{X} d \) to 0.2 is presented in Table 1.

### Scaling Model for \( \varepsilon_{sf} \)

The approach taken in this paper is to assume that the redistribution process attributable to secondary flow has a nature similar to turbulent mixing. Based on this premise, an eddy viscosity concept is used to define an effective viscosity coefficient due to secondary flow, \( \nu_{sf} \). The assumption is acceptable within the confines of the spanwise transport model, equations (1) and (2), even though secondary flow is an inviscid mechanism. The application of the eddy viscosity approximation and a Schmidt number of unity, allows \( \varepsilon_{sf} \) to be prescribed by a velocity scale and a length scale. The velocity scale is represented by the secondary flow kinetic energy \( \lambda \) and the length scale \( L_{sf} \), by some appropriate measure of the secondary flow vortices

\[ \varepsilon_{sf} = A_{sf} \sqrt{\lambda L_{sf}} \]  

where \( A_{sf} \) is a constant.

The secondary kinetic energy is determined by applying inviscid vortex theory to the uniform density flow through a rotating linear cascade. The analysis is based on that of Hawthorne (1955) and is contained in Appendix 1. If the velocity profile at inlet to the cascade is assumed to consist of a linear gradient with boundary layer thickness \( \delta_1 \) and a freestream velocity of \( V_1 \), the mean secondary kinetic energy is

\[ \lambda = \frac{V_1^2 \Delta^2}{h} \frac{\delta_1}{p} \frac{\sin (\alpha_1 - \beta_1) \cos \beta_1 + \cos \beta_2}{\cos (\alpha_1 - \beta_1) (\sin 2\beta_2 - \sin 2\beta_1) + \beta_2 - \beta_1} \]

where

\[ \Delta = \frac{-\sin (\alpha_1 - \beta_1) \cos \beta_1 + \cos \beta_2}{\cos (\alpha_1 - \beta_1) (\sin 2\beta_2 - \sin 2\beta_1) + \beta_2 - \beta_1} \]

and \( f(\tau) \) is a series expansion given in Appendix 1. It has the form

\[ p \overline{L} \]

shown in figure 1. This expression for \( \lambda \) is based on a stationary wall being upstream of the rotating cascade. A similar expression, representing the case of a rotating wall upstream of a stationary cascade is presented in Appendix 1. The two expressions (A1) and (A3) model the general cases of flow into a rotor and stator respectively.

The maximum possible size of the secondary flow vortex will be determined by the throat of the cascade. This can be approximated by \( p \) (figure A1). Substituting for \( \lambda, L_{sf} \) and non-dimensionalising gives

\[ \frac{\varepsilon_{sf}}{V_2 L_s} = A_{sf} \left( \frac{\Delta^2}{h} \frac{\delta_1}{p} \frac{f(\tau)}{p} \right) \frac{1/2}{L_s} \]

A similar analysis yields the effective diffusion coefficient \( \varepsilon_{sf} \) downstream of a stationary cascade. It has the exactly the same form as equation (7) except \( \Delta \) is determined by equation (A3) and the absolute flow angle at inlet \( \alpha_1 \) is replaced by the relative flow angle \( \beta_1 \). The experimental data presented in Part 1 were used to establish an appropriate value of \( A_{sf} \). The exit flow angles based on \( \cos^{-1} (o/p) \) of the two turbine builds were used as input to the Denton throughflow code to obtain the corresponding inlet angles. This analysis was performed about the third rotor of the LL turbine and the second rotor of the HL turbine. The values calculated at hub and tip were used to determine the parameters in equation (7) and hence for each build two separate values of \( \varepsilon_{sf} \). As the boundary layer thickness in the multistage environment is difficult to define and therefore measure, the peak value (0.05) of the function \( f(\tau) \) was used. The experimental data of figures 13 and 14, Part 1 show a degree of scatter but indicate that spanwise transport is fairly uniform across the span. The latter trend is reflected by the calculated values of \( \varepsilon_{sf} \) shown in Table 2 with \( A_{sf} \) set equal to 0.02.

### Table 1: Comparison of Estimated and Experimental Values of \( \frac{\varepsilon_d}{V^2 L_s} \)

<table>
<thead>
<tr>
<th>Turbine Build</th>
<th>Calculated</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>LL</td>
<td>0.0009</td>
<td>0.00007-0.0009</td>
</tr>
<tr>
<td>HL</td>
<td>0.0020</td>
<td>0.0018-0.0026</td>
</tr>
</tbody>
</table>

### Table 2: Comparison of Estimated and Experimental Values of \( \frac{\varepsilon_{sf}}{V^2 L_s} \)

<table>
<thead>
<tr>
<th>Turbine Build</th>
<th>Estimated Hub</th>
<th>Estimated Tip</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>LL</td>
<td>0.0027</td>
<td>0.0015</td>
<td>0.0016-0.0026</td>
</tr>
<tr>
<td>HL</td>
<td>0.0071</td>
<td>0.0051</td>
<td>0.0042-0.0056</td>
</tr>
</tbody>
</table>

...
IMPLEMENTATION OF SPANWISE TRANSPORT MODEL

From the preceding analysis it is possible to estimate $e_d$ and $e_{sf}$. These are average quantities and a method of distribution across the span has to be considered. In this study a linear distribution of $e_{sf}$ and a constant distribution of $e_d$ across the span were used. The local value of $e_{sf}$ was determined by applying equation (7) to the hub and casing regions of each blade row and assumed to extend from mid-chord of the particular blade row to mid-chord of the succeeding blade row.

In Part 1, it was shown that the stages upstream of the repeating stage experience a reduced level of freestream turbulence. A modified form of equation (2) is therefore suggested which allows the application of the scaling model to the initial stages of a multistage turbine

$$\epsilon = \left( e_d + e_{sf} \right) \tanh \left( \frac{z}{1.5 L_s} \right)$$  \hspace{1cm} (8)

The modification in equation (8) is somewhat arbitrary but can be justified by figure 2. This shows the variation of the estimated level of $\epsilon$ through the two turbine builds reported in Part 1 based on equation (8). The agreement with experimental data is considered acceptable.

The spanwise transport model is written into a subroutine that is called by the throughflow model after the inviscid distributions of stagnation enthalpy, entropy and angular momentum at each quasi-orthogonal have been calculated. The transport equation, equation (1), is discretised using finite differences and solved for the same total loss is approximately generated by each method. Additional loss is introduced by the throughflow model to account for shroud leakage; the distribution was constant for both cases. The throughflow model was run without spanwise transport using the two distribution methods. The predicted axial velocity profiles are compared with experimental data in figure 3b. The prescribed exit angle distributions which were based on measured data, ensure good predictions of the axial velocity in both cases. Although the loss distributions are significantly different, the effect on the axial velocity profile distribution is negligible except in the endwall regions. The introduction of spanwise transport has a minor influence on the axial velocity profile (figure 3b). The Reynolds number based on effective diffusion coefficient at rotor mid-span, axial velocity and stage length was Re=$330$. The spanwise distribution of efficiency (figure 3c) is strongly influenced however by both loss distribution and spanwise transport. By concentrating the secondary losses towards the endwall, the parabolic loss distribution lowers the spanwise efficiency in these locations. The linear loss distribution results in a flat efficiency distribution which is virtually independent of spanwise transport. The parabolic loss distribution and spanwise transport gives improved prediction over the region 30-100% span than does the linear distribution. The converse is true from hub to 30% span.

The redistribution process is irreversible itself and creates a certain amount of loss. As the scaling model described by equation (4) contains $\eta$, the stage efficiency, it is important that the coupling between the overall loss generation and spanwise transport is weak. The magnitude of this coupling was estimated by applying the parabolic loss distributions (figure 3a) as input to the throughflow model with and without spanwise transport. The efficiency, based on mass-averaged quantities at inlet and exit from the turbine, dropped by 0.3% from 91.0% when spanwise transport was included. Efficiencies based on throughflow calculations without spanwise transport can therefore be used as input to equation (4), to determine the appropriate level of $e_d$. 

APPLICATION OF THROUGHFLOW MODEL INCORPORATING SPANWISE TRANSPORT

One of the most important effects of spanwise transport in multistage compressors is the redistribution of entropy across the span (Adkins and Smith, 1982 and Gallimore, 1986). This must also be true to some extent in multistage turbines. An investigation was made using the LL turbine data described in Part 1 of this paper. The measured rotor and stator relative flow angles were used as input to the modified streamline curvature code. Two different methods were used to distribute the loss coefficient, determined by the built-in loss correlations, across the span. The resulting distributions for rotor 3 are shown in figure 3a. The first distribution is based on a linear distribution of secondary loss superimposed on the local profile loss and is the standard option available in the throughflow model (Denton, 1978). This method results in very flat loss distributions. The second method involves a parabolic distribution from endwall to mid-span of secondary loss superimposed on the local profile loss. The level of secondary loss coefficient was adjusted until the integrated value across the span was the same as in the first method. In this way the same total loss is approximately generated by each method. Additional loss is introduced by the throughflow model to account for shroud leakage; the distribution was constant for both cases. The throughflow model was run without spanwise transport using the two methods. The predicted axial velocity profiles are compared with experimental data in figure 3b. The prescribed exit angle distributions which were based on measured data, ensure good predictions of the axial velocity in both cases. Although the loss distributions are significantly different, the effect on the axial velocity profile distribution is negligible except in the endwall regions. The introduction of spanwise transport has a minor influence on the axial velocity profile (figure 3b). The Reynolds number based on effective diffusion coefficient at rotor mid-span, axial velocity and stage length was Re=$330$. The spanwise distribution of efficiency (figure 3c) is strongly influenced however by both loss distribution and spanwise transport. By concentrating the secondary losses towards the endwall, the parabolic loss distribution lowers the spanwise efficiency in these locations. The linear loss distribution results in a flat efficiency distribution which is virtually independent of spanwise transport. The parabolic loss distribution and spanwise transport gives improved prediction over the region 30-100% span than does the linear distribution. The converse is true from hub to 30% span.

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FIGURE 1: SERIES EXPANSION $\left( \frac{1}{P'} \right)$ AS A FUNCTION OF INLET BOUNDARY LAYER THICKNESS

FIGURE 2: ESTIMATION OF MID-SPAN EFFECTIVE DIFFUSION COEFFICIENT THROUGH LL AND HL TURBINES
The streamline curvature code has been applied to three other applications which have strong gradients across the annulus of either a scalar or stagnation temperature at inlet. The attenuation of these profiles was thought to be strongly influenced by spanwise transport. Joslyn and Dring (1990a, 1990b) experimentally studied the attenuation of an axisymmetric concentration profile in a one and a half stage low speed turbine, the profile at inlet simulating the spanwise temperature profile typically found at entry to a high pressure turbine. The turbine has a hub-to-tip ratio of 0.8 and an aspect ratio of approximately one. The maximum temperature profile is well predicted. If spanwise transport is set to zero, ReR=∞, the endwall temperatures are under-predicted by approximately 50 K and the mid-span region over-predicted by 30 K. By scaling the effective diffusion coefficient, equation (2), by 0.5 (ReR=700) the relative insensitivity of the calculations to the absolute level of spanwise transport is apparent.

The final test case is based on data obtained from a steam mixed-flow turbine situated in the Berkeley nuclear power plant (Curtis and Halliday, 1969). The two stage turbine is supplied steam from two sources: the inner part of the span, from the upstream high pressure turbine and the outer part by fresh steam obtained from a second heating loop directly from the reactor. In this turbine the maximum to minimum temperature variation at inlet is 33% of the mean temperature. At inlet the hub to tip ratio was 0.8 and the aspect ratio 2.2. The Denton throughflow code has the option of perfect gas properties or steam properties, the latter being used in this case. Design flow values were used as input to the throughflow model and the scaling models, the calculated efficiency as input to equation (4). The Reynolds number based on effective diffusion coefficient was ReR=420. From the experimental data a significant temperature profile still exists at exit from the second stage as seen in figure 5b. The calculated temperature profile with spanwise transport included shows improved agreement between experiment and calculation, the difference not being substantial. This reflects the reduced significance of spanwise transport in a higher aspect ratio turbine even though the Reynolds number ReR is approximately the same as in the previous test case.
CONCLUSIONS

The experimental evidence of the repeating stage as shown in Part 1 strongly suggests that spanwise transport is important and significant in low aspect ratio multistage turbines. Spanwise transport redistributes endwall losses towards mid-span by two mechanisms: turbulent diffusion and convection. It is necessary to model both of these mechanisms to generate the correct scale of transport. By incorporating a model of spanwise transport in a throughflow scheme it is possible to prescribe more realistic loss distributions across the span and thereby focus attention on where the losses are generated.

Calculations of the flow within three turbines showed that it is necessary to include the effects of spanwise transport in throughflow models in order to predict the correct attenuation of a spanwise temperature or scalar profile. Agreement with experimental data, considering the simplicity of the spanwise transport model was good.

The ability to predict this attenuation early in the design phase will aid the designer in considering spanwise distribution of cooling and thermal stressing.

ACKNOWLEDGEMENTS

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APPENDIX 1

The rate of generation of secondary kinetic energy is determined by considering inviscid vortex theory and applying it to the general case of uniform density flow through a rotating cascade. The analysis is based on that of Hawthorne (1955) and is considered in the intrinsic relative coordinate system $s^*, n^*, b^*$, shown in figure A1. If the relative Bernoulli surfaces are assumed cylindrical, the development of the absolute vorticity resolved along the relative streamline $s^*$ is given by (Horlock and Lakshminarayana, 1973)

$$\omega_{s1} = \omega_{n1} \sin (\alpha_1 - \beta_1)$$

$$\omega_{n1} = \omega_{n1} \cos (\alpha_1 - \beta_1)$$

The absolute vorticity resolved in the relative streamwise direction becomes

$$\omega_{s2} = \omega_{n1} \Delta$$

where subscript 1 and 2 refer to cascade inlet and exit stations. For a rotor, in the general case, the upstream wall will be stationary and the incoming vorticity, defined by $\omega = \omega_{n1} = \partial V_1 / \partial b$, has to be resolved into the relative streamwise $s^*$ and normal $n^*$ directions

$$\omega_{s1}^* = - \omega_{n1} \sin (\alpha_1 - \beta_1)$$

$$\omega_{n1}^* = \omega_{n1} \cos (\alpha_1 - \beta_1)$$

By defining a secondary stream function $\psi$ where $w_n = \partial \psi / \partial b^*$ and $w_b = - \partial \psi / \partial n^*$ a series solution can be found to the Poisson equation

$$\Delta = \psi_{s2} = \frac{\omega_{n1}^*}{\cos \beta_2}$$

$$\cos (\alpha_1 - \beta_1) \left( \sin 2\beta_2 - \sin 2\beta_1 \right) + \beta_2 - \beta_1 \right)$$

where $\psi = \sum_{k=1,3,5} \psi_k \sin k \pi b^*/p'$ and $\psi_k$ is given by a similar series. The boundary conditions, $\psi = 0$ where $n^* = 0$ and $n^* = p'$; $\psi = 0$ where $b^* = 0$ and $b^* = h$ define the complete solution for $\psi$

$$\psi = \sum_{k=1,3,5} \psi_k \sin k \pi b^*/p'$$

where for $0 < b^* < \delta$

$$\psi_k = \frac{4 \Delta p'^2 V_1}{(k \pi)^3 \delta} \left( 1 - e^{-k \pi b^*/p'} - e^{-k \pi b^*/p'} \sinh k \pi b^*/p' \right)$$

and for $\delta < b^* < h$

$$\psi_k = \frac{4 \Delta p'^2 V_1}{(k \pi)^3 \delta} e^{-k \pi b^*/p'} (\cosh k \pi b^*/p' - 1)$$

Using the series expansion for $\psi$ and its definition in terms of the secondary velocities $w_n^*$ and $w_b^*$, the mean kinetic energy $\lambda$ of the
The secondary flow is given by
\[ \lambda = \frac{p' h}{\int_0^h \int_0^1 (w_n^2 + w_b^2)dnb^*} \]

If the velocity profile is assumed to consist of a linear gradient with boundary layer thickness \( \delta_1 \) and a freestream velocity of \( V_1 \), \( \lambda \) becomes
\[ \lambda = \frac{V_1^2 \Delta^2 p'}{h f(\delta_1)} \]  
(A2)

where \( f(\delta_1) \) is a series expansion
\[ f(\delta_1) = 8 \left( \frac{\delta_1}{\delta_1} \right)^2 \sum_{k=1,3,5} \frac{1}{(k\pi)^5} \left( \frac{k\pi \delta_1/p'}{1 + e^{k\pi \delta_1/p'}} \right)^2 \cosh k\pi \delta_1/p' \]

\( f(\delta_1) \) has the form shown in figure 1.

A similar analysis can be performed for the stator. In the general case, the upstream wall will be rotating and the incoming absolute vorticity resolved in the intrinsic relative coordinate system is \( \omega = \omega_{n1}^* \). This has to be converted back into the absolute streamwise and normal directions
\[ \omega_{s1} = -\omega_{n1}^* \sin (\beta_1 - \alpha_1) \]
\[ \omega_{n1} = \omega_{n1}^* \cos (\beta_1 - \alpha_1) \]

The absolute vorticity becomes
\[ \omega_{s2} = \omega_{n1}^* \Delta \]

where
\[ \Delta = \frac{-\sin (\beta_1 - \alpha_1) \cos \alpha_1}{\cos \alpha_2} + \cos (\beta_1 - \alpha_1) \left( \frac{\sin 2\alpha_2 - \sin 2\alpha_1}{\cos \alpha_1 \cos \alpha_2} + \alpha_2 - \alpha_1 \right) \]

(A3)

The mean kinetic energy \( \lambda \) is
\[ \lambda = \frac{V_1^2 \Delta^2 p'}{h f(\delta_1)} \]

The series expansion \( f(\delta_1) \) is the same as in the rotor case.

\[ \lambda = \frac{W_1^2 \Delta^2 p'}{h f(\delta_1)} \]

(A4)