ABSTRACT

A model of the forced vibrations of a flexible, asymmetric and unbalanced shaft, supported by two magnetic bearings is derived to simulate the effect of different schemes of active control on shaft dynamic behaviour.

Simulation results were compared for several cases of single and multi-access bearing controls, rigid-body-mode only and rigid with flexible mode control, and linear and non-linear bearing responses. It is shown that the multi-access bearing response calculated from the known equation of the stable ROCL (Reduced Order Closed Loop) and based on the direct velocity-displacement feedback, provided the most precise shift in critical frequencies and also reasonable suppression of shaft vibration amplitudes.

The non-linear bearing design was also briefly discussed. The stability analysis showed that stability limits were influenced by more parameters in this case, but no particular advantages were observed in suppression of the vibration amplitudes as compared to the linear case.

Nomenclature

- \( \mu_0 \): magnetic permeability of the gap,
- \( A \): area of magnet pole,
- \( N \): number of coil turns (per pole) in magnet,
- \( I \): current in magnet coil,
- \([Gd], [Gv]\): coefficient or gain matrices, which comprise stiffness and damping coefficients, respectively, for each bearing actuator,
- \([Cd], [Cv]\): position/influence matrices of displacement and velocity sensors,
- \( \{q\} \): vector of modal coordinates,
- \([P], [P]\): modal and weighted modal matrix, respectively, (based on eigenvectors of undamped system),
- \([A]\): actuator position/influence matrix,
- \( [c] \): modal damping matrix of the system with no external forces,
- \( [A] \): eigenvalues of the undamped system,
- \( [F] \): vector of the external forces, (magnetic bearing response),
- \( m_i \): discrete shaft mass concentrated at the i-th location, (i=0,1,2),
- \( M_r \): shaft imbalance,
- \( K \): shaft stiffness,
- \( D \): external damping (usually negligibly small),
- \( F_i \): response of the respective bearing, (i = 0,2)
- \( A_i, \alpha_i \): vibration amplitude and phase angle for the i-th concentrated mass,
- \( z_k \): modal damping ratio for the k-th vibration mode,
- \( \pi \): index which indicates the order of modal truncation, (it is equal to the number of the primary modes).

1. INTRODUCTION

Magnetic bearings have been proven to be a novel and attractive form of shafts and rotors suspension. They possess some distinct advantages in that they do not generate wear and do not need lubrication. Their dynamic response can be designed and adjusted actively to control shaft vibrations in a wide range of operational speeds. They also have long life and great potential for shaft vibration suppression.

Disadvantages, however, include lack of sufficient field experience, unknown reliability over a long time, relatively high cost and required advanced automatic control and complex design. The design of a control system for a magnetic bearing is difficult and constitutes a challenging task which requires extensive experimental and theoretical efforts and investigations.

Dynamic models used to design the various components of the magnetic bearing and its controller, are quite numerous. They


comprise the classical problem of modeling the dynamic behaviour of the shaft, where the presence of the bearing is simulated as an external component to the system which has its own dynamic response. Complexity of the shaft models varies from the one-degree of freedom representation of spring-damper-mass to the multi-mass shaft described by matrix equations. Experimental work was mainly focused on testing different closed loop configurations, which furnished proper selection, optimization and tuning of the hardware parameters.

Some of the fundamental work done by Alliare, Humphris et al., [1,2,3,4,7,8,9,11], gave useful experimental insight into the contribution of the bearing stiffness and damping to the general rotor dynamic behaviour. Results of their experimental work with different circuit band widths were also compared in terms of the achieved stability regions. They have also proposed basic theoretical models for effective bearing stiffness and damping for a simple rotor configuration, with its mass lumped in a one-degree of freedom spring-mass system.

Quite extensive theoretical work, supported by experimental verification, has been developed by Schweitzer and Salm, [20,21,22,24], where a shaft-bearing system was represented in modal coordinates and a reduced order model was used to evaluate bearing response proportional to the direct velocity and displacement feedback. Similar approach has been presented in an excellent report by Strunce et al., [26] and by Lin et al., [14,15,16]. These papers gave also stability analysis and formulated stability conditions. Since the bearing force coefficients were calculated from the reduced-order model of the system, in order to ensure stability of the full-order rotor model special requirements had to be satisfied by the coefficient matrices in terms of matrix structure and number of controlled modes. Other design methods suitable for control design are discussed in [5,6,10,13,16,19,27].

Generally, the level of complexity of the shaft model determines, to some extent, the design of the bearing response. However, freedom in design of bearing response constitutes one of the greatest advantages in magnetic bearing application. Design of bearing response starts with the acceptance of certain control rule which specifies how the control current responsible for the magnitude of the bearing force, should react to signals measured by displacement and velocity sensors. Theoretically, any arbitrary function can be adopted as a design rule. The real limitations come actually from the capability and speed of the controller and cost of the required hardware. Also, highly sophisticated control strategy may not necessarily be the best solution to rotor vibration control. Therefore, the potential benefit resulting from the complicated bearing design has to be compromised and balanced with the cost of the proposed solution.

Review of the numerous reported experimental work on magnetic bearing concepts indicates that the linear bearing response, based on the direct velocity-displacement feedback, is the most popular technique employed in practical applications. Additionally, the linear bearing force is further restricted to only two terms: one which contributes to the system stiffness (displacement feedback) and the second which provides damping (velocity feedback). This suggests that difficulties associated with implementation of non-linear bearing control were not worth the potential benefits of this type of control. However, the justification of linear systems given in the literature is often based on experiments and experience. Few authors even mentioned the possibility of other than linear design of a bearing response.

The present paper provides simple numerical verification of the linear bearing response and comparison of effects of several different control scenarios on shaft dynamic behaviour. It also briefly discusses the influence of the non-linear bearing response on the stability of shaft operation.

Two separate models are used to simulate the dynamic behaviour of a shaft-bearing system. First, a general analytical model is derived for forced vibrations of an unbalanced, asymmetric, three-mass shaft supported by two magnetic bearings with collocated sensors. Second, for the assumed linear type of control the stable, reduced-order-closed-loop (ROCL) equation is accepted as the bearing design algorithm. For this algorithm the design objectives are specified in terms of the required amount of damping and shift in shaft natural frequencies which should result from the presence of the bearing and, consequently, the corresponding bearing force coefficients, i.e. stiffness and damping gains are calculated. This calculated bearing force is then used in the analytical model of the system to simulate behaviour of shaft controlled by the designed bearing force. The results showing the actually achieved amplitudes and critical frequencies of the shaft are compared with those specified as the objectives for the bearing force algorithm. Magnitudes of observed discrepancies can be recognized as quantitative measures of the spillover effects. Simulations are performed assuming single and multi-access bearing responses. Different modes are also selected for the direct control. Effectiveness of the considered types of control is compared in terms of suppression of shaft vibration and accuracy in the achieved shift in natural frequencies.

Finally, the non-linear bearing force is introduced into the derived analytical shaft-bearing model and the shaft vibrations are simulated and compared to the linear control case. No apparent benefits are observed in vibration suppression. However, brief analysis of the stability of the system shows that the non-linear type of control creates more possibilities to increase system stability limits, since they become dependent on larger number of parameters including rotational speed and shaft imbalance.

2. Linear Magnetic Bearing Response

In an active magnetic bearing system, the rotating shaft is suspended in the magnetic field created by stationary electromagnets. The position of shaft is dynamically corrected when the sensors detect any shaft departures from equilibrium (centerline) position. Magnets are usually working in attractive mode, i.e. the shaft displacement is corrected by the increased pull in the desired direction.

The electro-magnetic force exerted on a mass located at the distance x from the magnet is a following function of the mass position itself, x, and of the current I in the magnet coil:

\[ F = \frac{\mu_0 A N I^2}{2 x^2} \]  

(2.1)

If the shaft departs from its equilibrium position by distance \( \Delta x \), the control system should adjust the bearing response, by applying the proper correction \( \Delta I \) to the magnet control current. Assuming that the shaft displacement is small, the resulting corrective force, \( \Delta F \), can be estimated from the linearized Eq.(2.1):

\[ \Delta F = F(x_0 + \Delta x, I_0 + \Delta I) - F(x_0, I_0) = \frac{\mu_0 A N I_0^2}{2 x_0^2} \left( \Delta I - \frac{I_0}{x_0} \Delta x \right) \]  

(2.2)
where \( F(x_0,y_0) \) is the stationary force sufficient to support shaft weight in equilibrium position. The net force responsible for correction of shaft departure given by Eq.(2.2) is a function of two independent variables: shaft displacement \( \Delta x \), which can be only implicitly controlled, and the current \( \Delta I \) which can be changed directly and, therefore, becomes the subject of control design and optimization.

Assuming that changes in the magnet coil current are proportional to the local shaft displacements measured by all available sensors and also to the displacement velocities, which are either directly measured or calculated by differentiating the displacement signal in a controller, i.e.:

\[
\mathbf{I} \propto \begin{bmatrix} \{z\}_{\text{meas}} & \{\dot{z}\}_{\text{meas}} \end{bmatrix}
\]

(2.3)

and assuming negligible phase lags and rolloffs of the control circuit, it follows that the bearing response is proportional to the position and velocity feedbacks:

\[
F \propto \begin{bmatrix} \mathbf{Gd} & \mathbf{Cd} \end{bmatrix} \begin{bmatrix} \{z\}_{\text{meas}} \\ \{\dot{z}\}_{\text{meas}} \end{bmatrix}
\]

(2.4)

Design of the bearing response, which is now reduced to the determination of both gain matrices, should satisfy the following goals: i) to provide sufficient damping of the shaft vibrations; ii) to cause the required shift in natural frequencies and iii) to ensure stable shaft operation in a specified range of rotational speed. One of the methods to achieve this is based on the modal representation of shaft vibration, with the bearing response acting as an external force, \([24,26,27]\). Since this approach will be implemented further in the present model of shaft controlled vibrations, it is briefly summarized below:

The shaft equation in modal coordinates, suitable for the control design, has to be reduced to include only the primary vibration modes which are directly controlled:

\[
\{\ddot{q}\}_p + [c]_p \{\dot{q}\}_p + [\Lambda]_p \{q\}_p = [P]_p^T [A] \{F\}
\]

(2.5)

Substitution of the bearing force defined by Eq. (2.4) into the above shaft model results in the reduced-order-close-loop (ROCL) equation with the new overall stiffness and damping properties increased due to contribution of the bearing forces.

Once these new stiffness and damping of the system are specified as the design objectives, the resulting bearing gain matrices, \([\mathbf{Gd}]\) and \([\mathbf{Gv}]\), are calculated as follows, \([14,24,25]\):

\[
[\mathbf{Gd}] = ([P]_p^T [A])^T \left( [P]_p^T [A] [P]_p [A] \right)^{-1}
\]

\[
[\mathbf{Gv}] = ([P]_p^T [A])^T \left( [P]_p^T [A] [P]_p [A] \right)^{-1}
\]

(3.1)

Therefore, in order to ensure stability of the full-order model, additional conditions have to be satisfied in the gain calculations. References \([25,26,27,38,42]\) present good review and justification of these conditions in more details. What will be relevant to the present simulation is that, firstly, the reduced order model has to contain at least all rigid and undamped modes of vibration and, secondly, both augmented damping and stiffness matrices have to be positive definite. It follows that the total number of velocity/displacement sensors in the system has to be at least equal to the number of the desired controlled modes.

3. SHAFT-MAGNETIC BEARING MODEL

Consider the simple unbalanced asymmetric and flexible shaft supported by two magnetic bearings shown in Fig.1. Each bearing consists of two actuators placed in \(0-x\) and \(0-y\) planes with the corresponding collocated sensors. Shaft mass is concentrated at both ends corresponding to the bearing locations, and at the mid-span plane. The mathematical model of the unbalance-forced shaft vibration with no gyroscopic effects and negligible small cross-coupling, written in the complex coordinates \(z=x+jy\), represents the classical balance of forces:

- 'T'-disc: \( m_1 \dot{z}_1 + D_2 + K(z_1 - z_0) + K(z_1 - z_2) = M \omega^2 e^{i(\omega t)} \)

(3.1)

- '0'-mass: \( m_0 z_0 + K(z_0 - z_1) = F_0 \)

(3.2)

- '2'-mass: \( m_2 \dot{z}_2 + K(z_2 - z_1) = F_2 \)

(3.3)

The following particular solution of the differential equations (3.1)-(3.3) describes rotor synchronous forced vibrations:

\[
z_i = A_i \varphi_i(\omega t + \alpha_i) \quad \text{for} \; i = 0,1,2
\]

(3.4)

The bearing responses, \( F_0 \) and \( F_2 \), appear in the model (Eqs.3.2 and 3.3), and hence will obviously influence the solution which determines both the amplitudes and phase angles of rotor vibration.
Assuming the linear, multi-access bearing design, based on the shaft displacement/velocity measured by both sensors in each plane, with no cross-coupling between 0-x and 0-y planes, the bearing response takes similar form in both planes, i.e.:

\[ T_6 = (T_4 T_1)^2 + T_5^2 + 2T_1 T_4 T_5 \cos(\gamma_1 + \gamma_4 - \gamma_5) \]  
\[ T_5 = \sqrt{(X_1 X_{20})^2 + K^4 - 2K^2 X_1 X_{20} \cos(\phi_1 + \phi_{20})} \]  
\[ T_4 = \frac{X_2^2 + X_{20}^2 - 2X_2 X_{20} \cos(\phi_2 - \phi_{20})}{X_2^2 + X_{20}^2 - 2X_0 X_{30} \cos(\phi_0 - \phi_{30})} \]  
\[ T_1 = \sqrt{(X_0 X_1)^2 + K^4 - 2K^2 X_0 X_1 \cos(\phi_0 + \phi_1)} \]  

where:

\[ F = \begin{bmatrix} \{F_x\} \\ \{F_y\} \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ y_2 \end{bmatrix} \]  
(3.5)

where the gain matrices \([Gd]\) and \([Gv]\) are defined as:

\[ [Gd] = \begin{bmatrix} K_{Fx,x_0} & K_{Fx,x_2} \\ K_{Fx_2,x_0} & K_{Fx_2,x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  
(3.6a)

\[ [Gv] = \begin{bmatrix} D_{Fx,x_0} & D_{Fx,x_2} \\ D_{Fx_2,x_0} & D_{Fx_2,x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  
(3.6b)

The first subscript of each term in the gain matrix denotes the location of the actuator to which the particular gain applies, the second - the location of the sensor which provides measurement multiplied by the given gain, e.g. \(K_{Fx_2,x_0}\) is the stiffness coefficient which multiplied by the displacement measured by sensor located at 'x_0' will become part of the response of the actuator at 'x_2' location in 0-x plane. It follows, that for the single-access bearing design, only the terms on the main diagonal would be retained since response of each actuator is proportional only to the signal measured by its own sensor. The \([0]\)-terms in the second and third quarters of each matrix indicate the lack of cross-coupling between actuator forces in 0-x and 0-y planes.

### 3.1 Synchronous forced vibrations of unbalanced shaft supported by two multi-access magnetic bearings

Substituting the bearing response defined by Eqs.(3.5) and (3.6) into shaft dynamic model given by Eqs.(3.1)-(3.3) and solving it, the following relations are obtained for the amplitudes and phase angles for three concentrated shaft masses:

- '2' mass:
  \[ A_2 = \frac{KMw^2}{T_6} \quad \alpha_2 = -\gamma_6 \]  
(3.7)

- '0' mass:
  \[ A_0 = T_4A_2 \quad \alpha_0 = \alpha_2 + \gamma_4 \]  
(3.8)

- '1' disc:
  \[ A_1 = \frac{T_4A_2}{K} \quad \alpha_1 = \alpha_2 + \gamma_7 \]  
(3.9)

\[ X_0 = \sqrt{(K + K_{F_0,x_0} - m_0w^2)^2 + (\omega D_{F_0,x_0})^2} \]  
(3.21)

\[ X_2 = \sqrt{(K + K_{F_2,x_2} - m_0w^2)^2 + (\omega D_{F_2,x_2})^2} \]  
(3.23)

\[ \tan(\phi_0) = \frac{\omega D_{F_0,x_0}}{K + K_{F_0,x_0} - m_0w^2} \]  
(3.22)

\[ \tan(\phi_2) = \frac{\omega D_{F_2,x_2}}{K + K_{F_2,x_2} - m_0w^2} \]  
(3.24)
The above analytical solution given by Eqs.(3.7)-(3.31) describes the forced synchronous vibration of the unbalanced asymmetric shaft controlled by two multi-access bearings with collocated sensors. Bearing stiffness and damping coefficient appear explicitly in the model. Once the coefficient values are determined, shaft controlled vibration can be simulated; the resulting magnitude of vibration amplitudes and critical frequencies will provide assessment of the quality of control achieved with the given bearing design, as will be shown in Section 3.3 below.

It is worth noting, that the presented analytical solution, although complex in nature, has definite advantage as compared to a numerical solution, in that it demonstrates explicitly the effects of the shaft parameters such as mass, imbalance, stiffness, rotational speed, etc., and also the effects of bearing stiffness and damping, on the amplitudes and phase angles of the shaft concentrated masses. However, for a complex multi-mass shaft, the analytical solution is not attainable, and the numerical model is the only way to simulate shaft vibration. Numerical model can be easily developed based on the same principles as the existing numerous dynamic simulators for a shaft supported by conventional bearings, which comprise dynamic equations written in a matrix form for each of the shaft concentrated masses. The difference will be only in the magnetic bearing model, because while the stiffness and damping of the conventional bearing depends on the shaft deflection strictly at the bearing location, magnetic bearing response is modeled as proportional to the chosen local deflections at the sensor locations. However, since the numerical models are of a quantitative, not qualitative nature and do not provide the closed form solution, the analytical model showing a tangible relation among the important system parameters, seems to be better suited for the analysis and understanding of the problem.

3.2 Modal analysis of the shaft - determination of the linear bearing force coefficients

Stiffness and damping gains for the linear, multi-access bearing response will be calculated from the gain algorithm, given by Eqs.(2.6) and (2.7).

For gain algorithm the shaft eigenvectors and eigenvalues have to be known. Therefore, those parameters were estimated first from the free shaft model, represented by Eqs.(3.1)-(3.3), with no imbalance, damping and external forces present.

For this shaft with three degrees of freedom in each plane, the calculated three eigenvalues (natural frequencies), identical for 0-x and 0-y planes, and three corresponding eigenvectors (mode shapes) summarized in modal matrix $[\beta]$, are as follows:

$$\omega_1^2 = 0$$

$$\omega_2^2 = \frac{K}{2}\left[\frac{1}{m_0} + \frac{2}{m_1} + \frac{1}{\sqrt{\left(\frac{m_0}{m_2}\right)^2 + \left(\frac{2}{m_1}\right)^2}}\right]$$

$$\omega_3^2 = \frac{K}{2}\left[\frac{1}{m_0} + \frac{2}{m_2} + \frac{1}{\sqrt{\left(\frac{m_0}{m_1}\right)^2 + \left(\frac{2}{m_2}\right)^2}}\right]$$

$$[\beta] = \begin{bmatrix}
1 & 1 & 1 \\
1 & Y1 & Y3 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

where:

$$Y1 = \frac{1}{2} - m_0\left[\frac{1}{2m_0} + \frac{1}{2m_2} - \frac{1}{2}\sqrt{\left(\frac{1}{m_0} - \frac{1}{m_2}\right)^2 + \left(\frac{2}{m_1}\right)^2}\right]$$

$$Y2 = Y1 - m_0\left[\frac{1}{2m_0} + \frac{1}{2m_2} - \frac{1}{2}\sqrt{\left(\frac{1}{m_0} - \frac{1}{m_2}\right)^2 + \left(\frac{2}{m_1}\right)^2}\right]$$

$$Y3 = \frac{1}{2} - m_1\left[\frac{1}{2m_0} + \frac{1}{2m_2} + \frac{1}{2}\sqrt{\left(\frac{1}{m_0} - \frac{1}{m_2}\right)^2 + \left(\frac{2}{m_1}\right)^2}\right]$$

$$Y4 = Y3 - m_1\left[\frac{1}{2m_0} + \frac{1}{2m_2} + \frac{1}{2}\sqrt{\left(\frac{1}{m_0} - \frac{1}{m_2}\right)^2 + \left(\frac{2}{m_1}\right)^2}\right]$$

The first natural frequency and the mode shape are rigid mode parameters.
The numerical calculations carried out with the following shaft data: m_0=1.8kg, m_1 (disc)=68.0kg, m_2=2m_0=3.6kg, K=24000N/m, Mr=0.0036kgm, gave the natural frequencies of the shaft ω_1=83.67rad/s and ω_2=117.06rad/s, and the corresponding normalized mode shapes shown in Fig.2.

![Fig. 2: Shaft modes of vibration](image)

The amplitudes of the shaft free vibration are presented in Fig.3.

![Fig. 3: Amplitudes of undamped vibration of free three mass shaft](image)

It shows that each of the concentrated shaft masses experienced zero amplitude at a particular natural frequency, while other two masses were acting as an absolutely effective undamped absorbers.

With the calculated eigenvectors and eigenvalues, the design objectives for the bearing gains can now be specified in terms of the desired shifts in the natural frequencies and the required amount of damping.

Since the shaft-bearing system has only two position sensors in each plane, the number of the controlled primary modes has to be reduced to the rigid mode and the first flexible mode. The third flexible mode which is beyond control will certainly be affected by the bearing force, but in an unpredictable way. This will be demonstrated by means of the following two examples.

**Case I:**

It is required to achieve the following shift in frequencies and amount of damping for the controllable modes in each plane:

ω_1 = 0 rad/s \ 
ω_2 = 83.67 rad/s \ 
should be shifted to \ 
ω_1 = 5 rad/s \ 
ω_2 = 100 rad/s

It follows, that the additional stiffness resulting from the magnetic bearing presence is determined as:

\[ [\Lambda]^* = [\Lambda]_{req} - [\Lambda]_{exist} = \begin{bmatrix} 25 & 0 \\ 0 & 3000 \end{bmatrix} \]  
(3.35)

Additionally, the second design objective is equal modal damping ratio required for both controlled modes:

\[ z_k = 0.707 \]  
for \( k = 1,2 \)

Since the existing damping ratio is negligible, entire damping is practically provided by the bearing force and is calculated as the product of the required ratios and the respective natural frequencies:

\[ [c]^* = [c]_{req} - [c]_{exist} = 2([z]_{req} - [z]_{exist}) \begin{bmatrix} 7.07 & 0 \\ 0 & 14.14 \end{bmatrix} \]  
(3.36)

With both design objectives given by Eqs.(3.35) and (3.36) and the reduced weighted modal matrix \([P]^*\) calculated from the known eigenvectors and eigenvalues of the free shaft, the following gain matrices were obtained from Eqs.(2.6) and (2.7):

\[ [Gd] = \begin{bmatrix} 10815 & -9154 \\ -9154 & 9328 \end{bmatrix} \]  
(3.37)

\[ [Gv] = \begin{bmatrix} 864.1 & -394.3 \\ -394.3 & 443.5 \end{bmatrix} \]  
(3.38)
Both matrices are symmetric, as a consequence of the collocation condition. The fourth matrix quarters which contain stiffness and damping coefficients for the actuators in 0-y plane are identical to those for 0-x plane. This is an obvious consequence resulting from the assumed lack of cross-coupling and from the same design objectives required for both planes.

**Case II:**

In this case, the same shift in natural frequencies and the same damping ratio for the rigid mode are required as in the first case. However, a lower damping ratio $\zeta_2 = 0.11545$ is assumed for the flexible mode. Therefore, while the stiffness matrix remains the same, given by Eq.(3.35), the new damping matrix becomes:

$$[c'] = [c]_{eq} - [c]_{exist} = \begin{bmatrix} 7.07 & 0 \\ 0 & 23.09 \end{bmatrix} \quad (3.38)$$

The resulting second set of bearing gain matrices is:

$$[G_d] = \begin{bmatrix} 10815 & -9154 \\ -9154 & 9328 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.39)$$

$$[G_v] = \begin{bmatrix} 496.9 & -27.1 \\ -27.1 & 76.3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.40)$$

Comparison of the numerical results obtained in both examples shows no change in the calculated bearing stiffness coefficients, $[G_d]$, since the required shift in natural frequencies was the same in both cases; however, the different damping of the controlled modes specified for both cases resulted in substantial differences in the entries of $[G_v]$ matrices.

### 3.3 Verification of the bearing design objectives

The above calculation procedure of the bearing gains was based on the truncated modal representation of the shaft, with one elastic mode excluded from the control design. Therefore, the bearing design objectives specified as a required shift in the critical frequencies and damping of the primary modes will not be achieved precisely in the control of the full order shaft model. The highest uncontrollable elastic mode, truncated in the above analysis, will be also affected by the active control, with its amplitude damped and its frequency shifted. This, in turn, affects the critical frequencies and damping of the primary modes, causing discrepancies between the designed and actually achieved dynamic behaviour of the controlled modes; a phenomenon known by the spillover effect.

To estimate magnitude of spillover effects and to verify the accuracy of bearing design, the bearing force coefficients $[G_d]$ and $[G_v]$ calculated in both cases above, were substituted into the analytical model of shaft vibration, given by Eqs.(3.7) - (3.31). The shaft vibration amplitudes were determined in the entire frequency range, which covered all three modes. Results show the final effect of the bearing control with the visible discrepancies between the required and actually achieved design objectives, (Figs.4 through 7).
the actual shift in critical frequencies, first, no damping was introduced ([Gv])=0); the dotted curves show vibration of all three shaft concentrated masses controlled by the bearing force which possessed only the designed stiffness. The achieved frequency shift for the first rigid mode is precisely as has been designed, i.e. from 0 to 5 rad/s; the second controlled frequency was shifted to 93 rad/s, instead to 100 rad/s, as was required by the design objective; the third uncontrolled frequency has been shifted from its natural value of 117.06 rad/s to 143 rad/s. The solid lines show shaft vibrations controlled by the complete bearing force, including stiffness and damping terms. Vibration of the shaft at both ends have been significantly attenuated. The disc, however, experienced still large vibrations around the second critical frequency, but its amplitudes at the first and third criticals have been suppressed substantially.

Two other scenarios have also been considered and compared with the previously discussed full control results in Fig.5. In the first scenario, the number of the primary modes had been reduced only to the rigid mode. For this mode the required frequency shift and a damping ratio were as in the both previous cases, (i.e. \( \omega_1 = 5 \) rad/s, and \( z_1 = 0.707 \)). Thus, \([A]^*\) and \([c]^*\) matrices would be as follows:

\[
[A]^* = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \quad [c]^* = \begin{bmatrix} 0.707 & 0 \\ 0 & 0 \end{bmatrix}
\]

The second scenario was the single access control, where the actuator response was proportional to the signal measured only by its own sensor. The gain matrices for this single access case would contain only the terms on the main diagonals, obtained by summing all entries in rows in the respective multi-access matrices \([Gd]\) and \([Gv]\) in Eq.(3.37), i.e.

\[
[Gd] = \begin{bmatrix} 1661 & 0 \\ 0 & 174 \end{bmatrix} \quad [Gv] = \begin{bmatrix} 0 \\ 1861 & 0 \\ 0 & 174 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
[Gd] = \begin{bmatrix} 469.8 & 0 \\ 0 & 49.2 \end{bmatrix} \quad [Gv] = \begin{bmatrix} 0 \\ 469.8 & 0 \\ 0 & 49.2 \end{bmatrix}
\]

It means that the corresponding magnet responses, e.g. in 0-x plane, were:

\[ F_{x0} = (K_{Fx0,0} + K_{Fx0,x2})x_0 + (D_{Fx0,0} + D_{Fx0,x2})x_0 \]
\[ F_{x2} = (K_{Fx2,x0} + K_{Fx2,x2})x_2 + (D_{Fx2,x0} + D_{Fx2,x2})x_2 \]

The results presented in Fig.5 indicated that the best suppression of amplitudes at both ends of the shaft as well as the most precise shift in critical frequencies was provided by the designed multi-access control. The rigid mode control ensured very good suppression of the shaft vibration except for the heavier end of the shaft.

Certain similarity can also be observed between the single access and rigid mode controls - both types of control resulted in good suppression of the disc vibration but the amplitude of the heavier end of the shaft was excessive.

Results for the Case II are shown in Fig.6. As expected, with the same bearing stiffness coefficients as in the Case I, the shift in critical frequencies was also similar. The only difference was shown in the suppression of shaft vibration, where the disc amplitude was more attenuated as compared to Case I, but the amplitude of the heavier end of the shaft became higher.
described in Case I. However, the single access control was rather a pure numerical exercise, where it was assumed that the actuator received displacement/velocity signals only from the

This case has also been compared with the results obtained with the rigid mode and the single access controls in Fig.7. The concept of the rigid mode control was the same as the one

Fig. 6: Case II: Shaft vibration controlled by multi-access bearings

Fig. 7: Case II: Comparison of different control scenarios
sensor at the opposite end of the shaft, so the bearing forces, e.g. in 0-x plane, become:

\[ F_{x0} = (K_{Fx0,x0} + K_{Fx2,x0})x_2 + (D_{Fx0,x0} + D_{Fx2,x0})\dot{x}_2 \quad \text{and} \]
\[ F_{x2} = (K_{Fx2,x2} + K_{Fx2,x0})x_0 + (D_{Fx2,x0} + D_{Fx2,x2})\dot{x}_0 \]

The gain matrices in this single-access scenario had only off-diagonal terms which were obtained, analogously to the conventional single-access control considered in Case I, by summing all the entries in rows in the respective multi-access gain matrices given by Eq.(3.39).

The results shown in Fig.7 also support conclusions drawn from the first case. The designed multi-access bearings provided the objective and ensured good control of shaft end vibration; however, suppression of disc vibrations was not effective. Rigid mode control provided sufficient damping, but again, the heavier end of the shaft experienced higher vibrations. As may also be expected, the assumed single access control based on the measurement received from the sensor opposite to the actuator, allowed for extreme amplitudes at both shaft ends; rather surprisingly, however, the disc maintained quite stable position in this purely hypothetical case.

4. NON-LINEAR BEARING RESPONSE

In all of the simulated cases presented above, regardless of the number of controlled modes, single or multi-access type of control, the magnetic bearing response remained a linear function of displacement and velocity. As has already been mentioned in the introduction, the non-linear bearing design rarely deserves attention, since it brings severe numerical and functional difficulties in the theoretical modeling and in the practical implementation.

To assess the characteristics of a non-linear bearing system, the following numerical exercise was carried out introducing the non-linear terms into the bearing force model derived analytically in Section 3.1.

Assume that the bearing force is single-access and the damping term includes a non-linear arbitrary function, \( \varphi \), of the measured respective displacement amplitude, i.e.:

\[ \{F\} = \{GdIz\}_{\text{meas}} + \{GV\} \begin{bmatrix} \varphi (|z|_{\text{meas}}) \end{bmatrix} \{z\}_{\text{meas}} \]  (4.1)

Substitution of the above force definition into the shaft vibration model given by Eqs.(3.1)-(3.3), complicates it to the extend that the explicit solution is impossible. Even with the further simplifying assumption of shaft symmetry, \((m_0 = m_2)\), phase angles and shaft amplitudes had to be calculated implicitly from the model.

Example calculations were carried out assuming two different non-linear terms in bearing response:

\[ \varphi (|z|_{\text{meas}}) \propto |z|_{\text{meas}}^3 \quad \text{and} \quad \varphi (|z|_{\text{meas}}) \propto |z|_{\text{meas}}^4 \]

Comparison of the results with the corresponding linear control case show similar shifts in critical frequencies for both bearing designs. It was rather expected, since the stiffness terms were identical. Non-linear force was responding more effectively in the vicinity of the shaft critical frequencies, but no apparent benefits were observed in terms of amplitude suppression as compared to linear force. Generally, the character of the obtained damping effects was extremely similar to the properly adjusted suppression due to the linear bearing force.

4.1 Stability analysis

Contrary to the simulation results with no major differences in the achieved vibration suppression and shift in critical frequencies, analysis of shaft stability shows qualitative difference between stability regimes of shaft controlled by non-linear and linear bearing force. The presence of non-linear damping term in bearing response not necessarily increases the stability limits, but it introduces new parameters into shaft motion characteristic equation, which can affect stability range. Those parameters were not able to alter stability boundaries of shaft supported by bearing with linear response. This will be illustrated below.

Stability analysis of the synchronous rotor vibrations and derivation of the rotor characteristic equation was carried out, adopting the classical perturbation method [17]. The variational real variable \( a(t) \) was introduced as the perturbation of rotor vibration amplitude in the following way:

\[ z_i = [A_i + a_i(t)] e^{i (\omega t + \alpha_i)} \qquad i = 0, 1, 2 \]  (4.2)

Consequently, this perturbation had to appear also as an argument in the non-linear part of the bearing response:

\[ \varphi (|z|) = \varphi (A + a(t)) = \varphi (A) + \left( \frac{d \varphi}{d A} \right) a(t) \]  (4.3)

Substitution of the above "perturbed" coordinates defined by Eqs.(4.2) and (4.3) into the shaft motion Eqs.(3.1)-(3.3) and omission of terms \( e^{i \omega t} \), results in the following set of equations which describes behaviour of the perturbation in time:

\[ '1'-\text{disc:} \quad m_1 (\ddot{a}_1 + 2j \omega \dot{a}_1 - \omega^2 a_1) + D(\dot{a}_1 + j \omega a_1) \]
\[ + 2K_a (K_{12} a_1 - K_{12} a_2)(\alpha_1 - \alpha_2) = 0 \]  (4.4)

\[ '0'-\text{mass:} \quad m_0 (\ddot{a}_0 + 2j \omega \dot{a}_0 - \omega^2 a_0) + [D_{Fz0,20} + \varphi (A_0)](\dot{a}_0 + j \omega a_0) \]
\[ + \left( \frac{d \varphi}{d A_0} \right) a_0 A_0 + (K + K_{Fz0,20}) a_2 - K_{12} (\alpha_1 - \alpha_2) = 0 \]  (4.5)

\[ '2'-\text{mass:} \quad m_2 (\ddot{a}_2 + 2j \omega \dot{a}_2 - \omega^2 a_2) + [D_{Fz2,22} + \varphi (A_2)](\dot{a}_2 + j \omega a_2) \]
\[ + \left( \frac{d \varphi}{d A_2} \right) a_2 A_2 + (K + K_{Fz2,22}) a_2 - K_{12} (\alpha_1 - \alpha_2) = 0 \]  (4.6)

Solution of these perturbation equations takes the form:

\[ a_i = E_i e^{i(\omega t + \alpha_i)} \qquad i = 0, 1, 2 \]  (4.7)
where $E$ - real or complex constant.

Substitution of Eq.(4.7) into Eqs.(4.3)-(4.6) leads to the following characteristic equation of shaft motion:

$$
(m_0^2 + Ds + 2K)m_s^2 + (K + K_{Fz0,20})m_s + (D_{Fz2,22} + \varphi_2(A_2))s + \left(\frac{d\varphi_2}{dA_2}\right)\omega A_0 + K + K_{Fz0,20} + K + K_{Fz2,22} = 0
$$

(4.8)

The numerical analysis of the roots of Eq.(4.8) defines the characteristic equation of shaft motion:

$$
K^2 m_0^2 + (K + K_{Fz0,20}) + [D_{Fz2,22} + \varphi_2(A_2)]s + \left(\frac{d\varphi_2}{dA_2}\right)\omega A_0 + \left(\frac{d\varphi_0}{dA_0}\right)\omega A_0 = 0
$$

(4.8)

Analyzing the derived equation it is worth to notice that assuming the bearing force as linear, ($\{\varphi\} = 0$), and, accordingly, equating the non-linear terms and their derivatives to zero, the characteristic Eq.(4.8) becomes the characteristic equation for the purely rotational shaft motion.

Without calculating the roots of Eq.(4.8), it shows that the presence of the non-linear terms in bearing response introduced new parameters into the characteristic equation: rotational speed of shaft $\omega$, and vibration amplitudes $A_i$ ($i=0,1,2$) of all concentrated masses which, in turn, depend on the magnitude of shaft imbalance. It opens more possibilities to influence the stability limits as compared to the linear force case, where only stiffness and mass of the shaft, and bearing properties (stiffness and damping gains) could affect shaft stable operational range.

The above observation is analogous to the one described for fluid bearing [17,18], where the presence of the non-linear components of fluid forces made rotor stability dependent on the imbalance and a wider region of shaft stable operation was obtained simple by increasing imbalance of the shaft.

5. Conclusions

Mathematical model of three mass shaft supported by two magnetic bearings with collocated sensors has been derived to simulate vibration control provided by the differently designed magnetic bearing force.

In this analysis, the linear bearing response has been assumed, with the stiffness and damping coefficients calculated from the stable ROCL equation. The multi-access bearing response was verified in terms of the precision in the achieved shift in critical frequencies and suppression of the amplitudes of rigid and flexible modes. Results were compared with the multi-access control of the rigid mode only, and with the single access control.

It was demonstrated that the reasonable control of shaft end vibration and the most precise shift in critical frequencies was provided by multi-access bearings. This design also satisfied stability requirements in that the bearing gain matrices must be positive definite and symmetric. Single access bearings gave good attenuation of the middle disc mass, but poor suppression of the amplitude of the heavier end of the shaft. Also, with the considered single access control, the desired shifts in critical frequencies were not attained. Finally, effect of the single access control appeared to be similar to the case of multi-access control of the rigid mode only, with two other residual flexible modes beyond control.

Non-linear control provided by the bearing force with the linear stiffness but non-linear damping part was also briefly analyzed and compared to the linear control results. No apparent benefits were observed in suppression of shaft vibration or shift in frequencies. However, stability analysis showed that with the non-linear type of control, the limits of stable shaft operation can be influenced by additional parameters such as imbalance and shaft rotational speed which had no effect on stability of the shaft operating with the conventional linear bearings.

6. References


