EIGENVECTOR DERIVATIVES OF GENERALIZED NONDEFECTIVE EIGENPROBLEMS WITH REPEATED EIGENVALUES

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ABSTRACT
A new method is presented for computation of eigenvalue and eigenvector derivatives associated with repeated eigenvalues of the generalized nondefective eigenproblem. This approach is an extension of recent work by Dailey and by Juang et al. and is applicable to symmetric or nonsymmetric systems. The extended phases read as follows. The differentiable eigenvectors and their derivatives associated with repeated eigenvalues are determined for generalized eigenproblem, requiring the knowledge of only those eigenvectors to be differentiated. Moreover, formulations for computing eigenvector derivatives have been presented covering the case where multi-groups of repeated first eigenvalue derivatives occur. Numerical Examples are given to demonstrate the effectiveness of the proposed method.

INTRODUCTION
Eigenvector derivatives are very useful in many fields of structural dynamics, such as system identification, structural control and optimization. Calculation methods for eigenvector derivatives associated with distinct eigenvalues have been developed by many researchers (Fox and Kapoor, 1968, Rogers, 1970, Rudisill and Chu, 1975, and Nelson, 1975). The increased effectiveness of these methods have made it possible to calculate the sensitivity of eigendata to changes in system parameters.

The computation problem of eigenvalue and eigenvector derivatives associated with repeated eigenvalues is much more complicated and difficult due to the nonuniqueness of the eigenvectors. Mills-Curran (1986) and Lim et al. (1989) have presented two algorithms for computing the eigenvector derivatives with distinct eigenvalue derivatives. Wei (1992) proposed another method using generalized inverse, but a complete solution has not been presented. Juang et al. (1989) developed a novel approach to investigate the existence of differentiable eigenvectors for a nondefective matrix that may have repeated eigenvalues and provided the formulations for computation of the eigenvector derivatives of the standard eigenproblem using the modal expansion technique, which, therefore, requires the complete solution of the eigenproblem. However, the formulations for calculating the eigenvector derivatives for the case in which repeated first eigenvalue derivatives occur are incomplete since the eigenvector derivatives associated with the rest distinct eigenvalue derivatives have not been shown.

On the other hand, Dailey (1989) improved Ojalvo's method (1986) and provided a powerful method for analyzing and computing the eigenvector derivatives of generalized eigenvalue problems with symmetric matrices. Unfortunately, this method is partially incorrect. In the paper, Dailey believed that the eigenvector derivatives are not unique when repeated eigenvalue derivatives occur and that "this provides a free parameter to set arbitrarily". He further suggested that "the simplest choice is to set c_i = 0.5q_i whenever \( \lambda_i = \lambda_i'^{-1} \). This is generally incorrect, except for the special case in which any higher eigenvalue derivatives are repeated, or in mechanical terminology as pointed out by Dailey (1989) "the parameter p affects the degenerate modes equally or not at all". In practice, however, it is also possible that one of the higher, but finite, derivatives of the eigenvalues becomes distinct when the first eigenvalue derivatives are repeated. It will be demonstrated in this paper that in this case the eigenvector derivatives are still unique.

An novel algorithm is provided in this paper for computing the derivatives of eigenvalues and eigenvectors for nondefective generalized eigenvalue problems with repeated eigenvalues. This algorithm requires the knowledge of only those eigenvectors that are to be differentiated. In this paper a new method is developed for calculating the particular solutions of the eigenvector derivatives based on the theory of linear algebraic equations, which leads to considerable mathematical simplification. Moreover, complete formulations are provided for computing the eigenvalue and eigenvector derivatives for the case where different repeated first eigenvalue derivatives occur. The general formulations for higher derivatives of the eigenvectors are also provided, which makes it possible to deal with the case where more complicated patterns of eigenvalue derivatives occur.

EXPRESSION OF THE PROBLEM
Consider the right and left nondefective eigenvalue problems for a \( N \) degrees of freedom structure
\[
K(p)\Phi_R(p) = M(p)\Phi_R(p)\Omega(p)
\]
(1)
\[
K^T(p)\Phi_L(p) = M^T(p)\Phi_L(p)\Omega(p)
\]
(2)
with biorthonormalization relation
\[
\Phi_L^T(p)M(p)\Phi_R(p) = I_N
\]
(3)

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where $K(p)$ and $M(p)$ are the stiffness and mass matrices, respectively, whose elements are analytic functions of the real scalar parameter $p$ in a neighborhood of $p_0$. As is usually the case in structural dynamics, it is assumed that $M(p)$ is symmetric and nonsingular. It may be noticed that on the nonsingularity assumption of the mass matrix the eigenproblems of Eqs. (1) and (2) can be transformed into the standard eigenvalue problems studied by Juang et al. (1989). However, calculations of the inverse of $M(p)$ and the derivatives, especially higher derivatives, of $K(p)M(p)^{-1}$ are not easy tasks. Hence, it is considered to be convenient to develop computational methods for the eigenvector derivatives of generalized eigenvalue problems.

The right and left eigenvector matrices in Eqs. (1)-(3) are designated as

$$
\Phi_R(p) = [Z_R(p) \ Y_R(p)] \quad \Phi_L(p) = [Z_L(p) \ Y_L(p)]
$$

and the eigenvalue matrix is defined as

$$
\Omega(p) = \text{diag}(\Lambda(p) \ \Delta(p))
$$

It is further assumed that

$$
K(p) \rightarrow K_0, \quad M(p) \rightarrow M_0, \quad \Omega(p) \rightarrow \Omega_0 = \text{diag}(\Lambda \ \Delta)
$$

$$
\Phi_R(p) \rightarrow \Phi_R = [Z_R \ Y_R], \quad \Phi_L(p) \rightarrow \Phi_L = [Z_L \ Y_L]
$$

$$(K(p) - \lambda_0 M(p) )X_R = 0, \quad (K(p) - \lambda_0 M(p) )X_L = 0 \quad (8,9)
$$

$$
X_R^T M_0 X_R = I_n \quad (10)
$$

which have a eigenvalue $\lambda_0$ at $p = p_0$ with multiplicity $n$, i.e., $\Lambda = \lambda_0 I_n$. Here $I_n$ is an $n \times n$ identity matrix.

Note that since the eigenvectors for a repeated eigenvalue are not unique, it is not known whether or not the eigenvector solutions associated with $\lambda_0$, denoted as $X_R$ and $X_L$, which also satisfy

are continuous and differentiable. And more generally they are not. However, since the column vectors of $X_L$ and $X_R$, respectively, span a left eigenspace and a right eigenspace for the eigenvalue $\lambda_0$, the differentiable eigenvectors can be expressed as

$$
Z_R = X_R \alpha_R, \quad Z_L = X_L \alpha_L \quad (11)
$$

Hence, it is of primary importance to find out the particular coefficient matrices $\alpha_R$ and $\alpha_L$ for determination of the differentiable eigenvectors and to verify the uniqueness of the column vectors in $Z_R$ and $Z_L$ that are independent on the choice of the eigenvectors of Eqs. (8)-(10).

**EIGENVECTOR DERIVATIVES**

Take the partial derivatives of Eq. (5) with respect to parameter $p$ and let $p \rightarrow p_0$, we obtain

$$
K_Z R \phi_i = F_i, \quad (12)
$$

where

$$
K = K_0 - \lambda_0 M_0, \quad K^{(i)} = K^{(i)} - \lambda_0 M^{(i)}
$$

$$
F_i = \sum_{k=1}^{i} C_i^k (\sum_{j=1}^{k} C_j^{k-i} M^{(k-j)} \lambda_0^{(j)} - K^{(j)} Z_R^{(k-j)})
$$

$$
C_i^k = \frac{i!}{k!(i-k)!} \quad (14)
$$

It can be proved (see Appendix A) that Eq. (12) is solvable if and only if

$$
Z_L^T F_i = 0 \quad (16)
$$

Substitution of Eq. (14) into Eq. (16) yields

$$
\Lambda^{(i)} = -Z_L^T g_i, \quad i = 1, 2, 3, \ldots \quad (17)
$$

where

$$
g_i = F_i - M_0 Z_R \Lambda^{(i)} \quad (18)
$$

is called the $i$-th order source excitation. Premultiplying Eq. (17) by $M_0 Z_R$ and substituting the resulting equation into Eq. (18) gives

$$
F_i = (I - M_0 Z_R Z_L^T) g_i = P_R g_i, \quad i = 1, 2, \ldots \quad (19)
$$

Here, a combination of Eq. (12) and (19) results in

$$
K Z_R^{(i)} = P_R g_i, \quad i = 1, 2, \ldots \quad (21)
$$

For convenience, we employ a frequency-shifted system, whose stiffness and mass matrices are $K = K_0 - \lambda_0 M_0$ and $M_0$. Then the shifted eigenvalue problem are

$$
K \Phi_R = M_0 \Phi_R \Xi, \quad \Phi_R = M_0 \Phi_L \Xi \quad (22)
$$

$$
\Xi = \text{diag}(0_{n \times n}, \Delta - \lambda_0 I_n) \quad (22a)
$$

The eigensolutions of this system can be written as

$$
\Phi_R = [Z_R \ \ Y_R], \quad \Phi_L = [Z_L \ \ Y_L] \quad (23)
$$

Here $r = N - n$, and equals to the rank of $K$. It can be seen from Eq. (23) that the column vectors of matrix $Z_R$, become the zero frequency modes, or called rigid body modes, of the frequency-shifted system, and that the eigenspaces spanned by the column vectors of $Z_R$ and $Y_R$ are the nullspace, denoted as $N(K)$, and the complementary space, denoted as $R(K)$, of $K$.

**EIGENVECTORS DERIVATIVES**

Generally speaking, the eigenvector derivatives can be expressed as a linear combination of the complete set of the eigenvectors of the system

$$
Z_R^{(i)} = [Z_R \ \ Y_R] (s_i) \equiv W_i + Z_R s_i, \quad i = 1, 2, \ldots \quad (24)
$$
where $s_1$ and $q_i$ are arbitrary coefficient matrices, and $W_i$, which is defined equal to $Y_R \mathbf{q}_i$, represents the component of $Z_W^{(i)}$ in $R(\mathbf{K})$. Such a representation of $W_i$ has been used by Juang et al. (1989) to determine the eigenvector derivatives. Since for large structural systems usually only a truncated set of mode shapes are computed, it is more practical and reasonable to express $W_i$ by $Z_R$.

In mathematical terminology (Lancaster, 1985), the solution set of Eq.(21) has the form of $W_i^* + N(\mathbf{K})$, and the general solutions can be expressed as

$$Z_W^{(i)} = W_i^* + Z_R c_i \quad (25)$$

where $W_i^*$ is a particular solution of Eq.(21) and $c_i$ is any $n$ by $n$ matrix. Since rank($\mathbf{K}$) = $r$ for a eigenvalue $\lambda_0$ with multiplicity $n$, $\mathbf{K}$ can then be reordered of the form

$$\mathbf{R}\mathbf{Q} = \begin{pmatrix} \mathbf{K}_{rr} & \mathbf{K}_{rs} \\ \mathbf{K}_{sr} & \mathbf{K}_{ss} \end{pmatrix}$$

In this equation $\mathbf{R}$ and $\mathbf{Q}$ are the transition matrices, and $\mathbf{K}_{rr}$ is a nonsingular submatrix of $\mathbf{K}$ and should be selected to be well conditioned. By using this equation, $W_i$ can be solved from Eq.(21)

$$W_i^* = \mathbf{R} \begin{pmatrix} \mathbf{K}_{rr}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{Q}\mathbf{F}_i \equiv \mathbf{G}_i \mathbf{F}_i \quad (26)$$

Observe that the particular solution determined by Eq.(26) depends on the choice of $\mathbf{K}_{rr}$. Nevertheless, it is of great importance to note that the different selection of the particular solution of linear algebraic equations does not change the solution set. As a result, both Eq.(24) and (25) are the formulations for the general solutions of Eq.(21). Hence, it is possible to determine the particular solution $W_i$ in Eq.(24) through Eq.(25). In fact, such a solution can be obtained by imposing the biorthonormal condition

$$Z_L^T \mathbf{M}_0 (W_i^* + Z_R c_i) = 0 \quad (27a)$$

which will be satisfied if we choose

$$c_i = -Z_L^T \mathbf{M}_0 W_i^* \quad (27a)$$

In other words, if $c_i$ is so chosen as in Eq.(27a), the solution given by Eq.(25) will be in $R(\mathbf{K})$. It is clear that this solution is just the particular solution required in Eq.(24) since Eq.(27) has been used. Therefore, substituting Eq.(27a) into Eq.(25) yields

$$W_i = \mathbf{P}_L S_i (Y_R a_l) \quad (28)$$

where

$$\mathbf{S}_i = \mathbf{P}_L^T \mathbf{G}_i \quad (28a)$$

$$\mathbf{P}_L = \mathbf{I} - \mathbf{M}_0 Z_L Z_L^T = \mathbf{I} - \mathbf{M}_0 X_L X_R^T \quad (28b)$$

The matrices $\mathbf{G}_i$ and $\mathbf{P}_L$ are referred to as the 'elastic' flexibility matrix (Craig, 1981) and the left projection matrix. Note that although the matrix $\mathbf{G}$ given by Eq.(25) is arbitrary in the sense of its dependence on the choice of $\mathbf{K}_{rr}$, it can be verified the 'elastic' flexibility matrix $\mathbf{G}_i$ is unique. Note also that the matrix $W_i$ that is defined in Eq.(26) can be calculated by using only the eigenvectors of concern, without requiring the knowledge of any other eigenpairs of the structure.

On the other hand, it is not difficult to notice that substituting Eq.(22) into Eq.(21) will yield

$$\mathbf{G}_i = Y_R (\Delta - \lambda_0 \mathbf{L})^{-1} Y_L^T \quad (29)$$

which indicates that $\mathbf{G}_i$ can also be calculated by $Y_R$ and $Y_L$, the 'elastic' modes of the frequency-shifted system. As a matter of fact, it has been demonstrated (not shown in this paper) that substitution of Eq.(29) and unit mass matrix into the formulations in the next section for computing the eigenvalue and eigenvector derivatives will result in the same formulations as those given by Juang et al. (1989). However, calculation of $\mathbf{G}_i$ by Eq.(28) requires less computational time.

**FORMULATIONS FOR COMPUTATION OF EIGENVALUE AND EIGENVECTOR DERIVATIVES**

From Eqs.(14)(17) and (18), we have

$$\Lambda^{(j)} = Z_L^T \mathbf{K}^{(j)} Z_R \quad (30)$$

Since $Z_L$ and $Z_R$ are normalized such that $\alpha_L^T \mathbf{Q}_R = \mathbf{I}$ and $\alpha_L$ is nonsingular, substitution of Eq.(11) into Eq.(30) yields

$$D_1 \alpha_R = \alpha_R A^{(j)} \quad (31)$$

where

$$D_1 = X_L^T \mathbf{K}^{(j)} X_R \quad (31a)$$

The first eigenvalue derivatives can be determined by eigenproblem (31). The formulations for calculation of $Z_W^{(j)}$ will be discussed as follows.

**Distinct Eigenvalue Derivatives**

For distinct eigenvalue derivatives

$$\lambda_j^{(i)} \neq \lambda_k^{(i)} \quad j \neq k ; \ j, k = 1, 2, \ldots, n \quad (32)$$

The solution of Eq.(31) will give the unique eigenvector matrix $\mathbf{Q}_R$, which can further determine $Z_R$ through Eq.(11). It can be proved that $Z_R$ is also unique, independent on the choice of eigenvectors of Eqs.(8)-(10). Consequently, it can be obtained

$$Z_L = X_L \alpha_R X_R^T$$

by using $\alpha_L^T \mathbf{Q}_R = \mathbf{I}_n$. Thus, the component of $Z_R^{(i)}$ in $R(\mathbf{K})$ can be determined using Eqs.(28), (18) and (14)

$$W_i = -G_i \mathbf{K}^{(i)} Z_R \quad (33)$$

To determine the coefficient matrix $s_1$, the second derivatives of eigenvalues are used. Let $i = 2$ for Eq.(14)(17) and (18), we get

$$\Lambda^{(2)} = -Z_L^T [2(\mathbf{M}_0 Z_R \Lambda^{(1)} - \mathbf{K}^{(1)} Z_R^{(1)} - \mathbf{K}^{(2)} Z_R + 2\mathbf{M}^{(1)} Z_R \Lambda^{(1)}]$$

Substituting $Z_R^{(1)} = W_i + Z_R s_1$ into this equation yields

$$s_1 \Lambda^{(1)} - \Lambda^{(1)} s_1 + 0.5 \Lambda^{(2)} = Z_L^T \mathbf{J} Z_R + \mathbf{L}_1 \Lambda^{(1)} \equiv \mathbf{U}_1 = (s_1)^T \quad (34a)$$

where

$$\mathbf{J} = 0.5 \mathbf{K}^{(2)} - \mathbf{K}^{(1)} \mathbf{G}_i \mathbf{K}^{(1)} \quad \mathbf{L}_1 = -Z_L^T \mathbf{M}^{(1)} Z_R \quad (34b)$$

The scalar form of Eq.(34a) can be written as

$$(\lambda_j^{(1)} - \lambda_i^{(1)}) s_{ij} + 0.5 \lambda_i^{(2)} \delta_{ij} = s_{ij}^{(1)} \quad (34c)$$
where the Kronecker delta is defined as

$$
\delta_{ij} = \begin{cases} 
1 & \text{if } i=j, \\
0 & \text{otherwise.}
\end{cases}
$$

Eq.(34c) results in

$$
\lambda_i^{(2)} = 2u_i^{ii} \quad (35a)
$$

$$
i^{ii} = \frac{u_i^{ii}}{\lambda_i^{(1)} - \lambda_i^{(1)}}, \quad i \neq j \quad (35b)
$$

Thus, the off-diagonal elements of $s_1$ have been determined in Eq.(35b). To further determine the diagonal elements of $s_1$, the normalization constraint

$$
Z_{Ri}^T M_0 Z_{Ri} s_1^i = -0.5Z_{Ri}^T M_1 Z_{Ri} - Z_{Ri}^T M_0 W_1^i \cong b_n \quad (36)
$$

Here $s_1^i$ and $W_1^i$ indicate the $i$th column vectors of $s_1$ and $W_1$, respectively. Hence, the diagonal elements of $s_1$ can be obtained

$$
s_1^i = b_n - \sum_{j=1, j \neq i}^{n} Z_{Ri}^T M_0 Z_{Rj} i^j, \quad i = 1, 2, \cdots , n \quad (37)
$$

Note that for the eigenproblem with nonsymmetric stiffness matrix, $Z_{Ri}^T M_0 Z_{Rj} \neq 0, i \neq j$. Hence, the diagonal elements of $s_1$ cannot be calculated before the off-diagonal elements are determined.

Repeated Eigenvalue Derivatives

In this section, we will consider the case in which repeated eigenvalue derivatives occur and are of the pattern

$$
\Lambda^{(1)} = \text{diag}(\Lambda_1^{(1)}, \Lambda_1^{(1)}, \cdots, \Lambda_k^{(1)}) \quad (37)
$$

where

$$
\Lambda_k^{(1)} = \lambda_k^{(1)} M_0, \quad m_k \geq 1, \quad k = 1, 2, \cdots , h
$$

$$
\lambda_1^{(1)} < \lambda_2^{(1)} < \cdots < \lambda_k^{(1)} \quad (38)
$$

$m_k (k = 1, 2, \cdots , h)$ is the multiplicity of the $k$-th group of repeated first eigenvalue derivatives, and $\sum_{k=1}^{h} m_k = n$. The corresponding right and left eigenvectors of $D_1$ are denoted as

$$
\Gamma_R = [ \gamma_{R1} \gamma_{R2} \cdots \gamma_{Rk} ]
$$

$$
\Gamma_L = [ \gamma_{L1} \gamma_{L2} \cdots \gamma_{La} ]
$$

where $\gamma_{Rk}$ and $\gamma_{Lk}$ represent the collections of the right and left eigenvectors associated with $\Lambda_k^{(1)} (k = 1, 2, \cdots , h)$. Due to the nonuniqueness of $\Gamma_R$ and $\gamma_{Rk}$, for any matrices $a_k$ and $b_k (k = 1, 2, \cdots , h)$, $\gamma_{Rk} a_k$ and $\gamma_{Lk} b_k$ are also the right and left eigenvectors of $D_1$. Hence, the differentiable eigenvectors cannot be identified yet. Clearly, only those $a_k$'s and $b_k$'s such that the eigenvectors of the initial system are differentiable at $p = p_0$ are of interest. Thus, Eq.(11) can be rewritten as

$$
Z_{Rk} = [ Z_{R1} Z_{R2} \cdots Z_{Ra} ] \quad (39)
$$

$$
Z_{Lk} = [ Z_{L1} Z_{L2} \cdots Z_{La} ] \quad (40)
$$

Substituting Eq.(39) and (40) into Eq.(34a) yields

$$
B_1 - B_2 + 0.5A^{(2)} = R \quad (41)
$$

where

$$
B_1 = \begin{pmatrix} 
s_{11} A_1^{(1)} & s_{12} A_1^{(1)} & \cdots & s_{1h} A_1^{(1)} \\
s_{21} A_1^{(1)} & s_{22} A_1^{(1)} & \cdots & s_{2h} A_1^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
 s_{11} A_h^{(1)} & s_{21} A_h^{(1)} & \cdots & s_{hh} A_h^{(1)}
\end{pmatrix}
$$

$$
B_2 = \begin{pmatrix} 
A_1^{(1)} s_{11} & A_1^{(1)} s_{12} & \cdots & A_1^{(1)} s_{1h} \\
A_2^{(1)} s_{21} & A_2^{(1)} s_{22} & \cdots & A_2^{(1)} s_{2h} \\
\vdots & \vdots & \ddots & \vdots \\
A_h^{(1)} s_{h1} & A_h^{(1)} s_{h2} & \cdots & A_h^{(1)} s_{hh}
\end{pmatrix}
$$

$$
A^{(2)} = \begin{pmatrix} 
A_1^{(2)} \\
A_2^{(2)} \\
\vdots \\
A_h^{(2)}
\end{pmatrix}
$$

$$
R = \begin{pmatrix} 
\tau_{11} & \tau_{12} & \cdots & \tau_{1h} \\
\tau_{21} & \tau_{22} & \cdots & \tau_{2h} \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{h1} & \tau_{h2} & \cdots & \tau_{hh}
\end{pmatrix}
$$

In the above equations

$$
r_{jk} = Z_{Lj}^T (J - \lambda_j^{(1)} M^{(1)}) Z_{Rk} \quad (42)
$$

and $s_{jk} (j, k = 1, 2, \cdots , h)$ are the $m_j$ by $m_k$ submatrices of $s_1$. Eq.(41) can be rewritten in the submatrix form

$$
s_{kk} A_1^{(1)} - A_1^{(1)} s_{kk} + 0.5A^{(2)} = r_{kk} \quad (43)
$$

$$
s_{jk} A_1^{(1)} - A_1^{(1)} s_{jk} = r_{jk} \quad j \neq k \quad (44)
$$

Noting that $s_{kk} A_1^{(1)} = A_1^{(1)} s_{kk}$, and that $b_{x}^T a_{k} = 1$ and $b_{x}^T$ is nonsingular, Eq.(43) can be rewritten as

$$
D_{21} a_k = s_{kk} A_1^{(2)} \quad k = 1, 2, \cdots , h \quad (45)
$$

where

$$
D_{21} = 2 \gamma_{L1} X_{L1}^T (J - \lambda_1^{(1)} M^{(1)}) X_{R1} \gamma_{R1} \quad (45a)
$$

For the case in which all the eigenvalues of Eq.(45) are distinct, the corresponding eigenvectors of Eq.(45) are unique. Again, $Z_L$, $Z_R$, and $b_k$ can be obtained through a procedure similar to the case of distinct first eigenvalue derivatives. Also, it can be proved that the column vectors of $Z_L$ and $Z_R$ are unique.

Therefore, the off-diagonal blocks of $s_1$ can be determined by Eq.(44)

$$
s_{jk} = \frac{Z_{Lj}^T (J - \lambda_j^{(1)} M^{(1)}) Z_{Rk}}{\lambda_j^{(1)} - \lambda_j^{(1)}} \quad j \neq k, \quad j, k = 1, 2, \cdots , h \quad (46)
$$

It remains to determine the diagonal submatrices of $s_1$. Clearly, the off-diagonal elements of the diagonal submatrices can not be obtained by Eq.(43).
Now using Eq.(17) for \( i = 3 \) and combining terms yields
\[
(s_2 \Lambda^{(1)} - \Lambda^{(2)} s_2) + \frac{1}{3} \Lambda^{(3)} + (s_1 \Lambda^{(2)} - \Lambda^{(3)} s_1)
= T + 2 L_1 (s_2 \Lambda^{(1)} - \Lambda^{(2)} s_2) + 2 (s_1 \Lambda^{(1)} - \Lambda^{(2)} s_1) s_2
\equiv U_2
\]
where
\[
T = ZL^T [K^{(1)} W_{22} + K^{(2)} W_1 + \frac{1}{3} K^{(3)} Z_R] - \left( 2 M^{(1)} W_1 + M^{(2)} Z_R \Lambda^{(1)} \right) L_1 \Lambda^{(2)}
\]
\[
W_{22} = G_k [2 (M^{(1)} Z_R + M_k W_1) \Lambda^{(1)} - K^{(2)} Z_R - 2 K^{(1)} W_1]
\]
\( s_2 \) is the coefficient matrix of \( Z_R \), which yields
\[
W_2 + Z_R s_2
\]
Partitioning Eq.(47), we get the diagonal submatrix equations, only which are of interest here
\[
\frac{s_{kk} \Lambda^{(2)} - \Lambda^{(3)} s_{kk} + 1}{3} \Lambda^{(3)} = T_{kk} + 2 \sum_{j=1, j \neq k} (\Lambda^{(3)} - \Lambda^{(2)}) s_{kj} + 1 \Lambda^{(2)} s_{jk} = U_{2k} = (w_{2k})
\]
\( \Lambda^{(3)} - \Lambda^{(2)} \) is the coefficient matrix of \( Z_R \), partitioned as \( s_{kk} \Lambda^{(2)} - \Lambda^{(3)} s_{kk} + 1 \Lambda^{(3)} = T_{kk} + 2 \sum_{j=1, j \neq k} (\Lambda^{(3)} - \Lambda^{(2)}) s_{kj} + 1 \Lambda^{(2)} s_{jk} = U_{2k} = (w_{2k})
\]
which yields
\[
\lambda_i^{(3)} = 3 \lambda_i^{(2)}
\]
\[
\lambda_{ij}^{(3)} = \frac{w_{ij}}{\lambda_i^{(2)}}, \quad j \neq i
\]
Thus, all the off-diagonal elements of \( s_1 \) have been determined, the diagonal ones can also be calculated using Eq.(37). It can be seen from Eqs.(46) to (52) and Eq.(37) that the coefficient matrix \( s_1 \) is unique. As a result, the eigenvector derivatives are also unique, since \( W_1 \) and \( Z_R \) are unique.

### SUMMARY OF COMPUTATIONAL STEPS

The complete algorithm is summarized as follows:

1. Compute the eigenvalues and eigenvectors of the eigenproblem as \( p = p_0 \), determine the multiplicity of the repeated eigenvalues.
2. Calculate \( G_\alpha \) using Eq.(28a).
3. Solve the eigenvalue problem, Eq.(31), to determine the first eigenvalue derivatives and the corresponding eigenvectors. If no repeated first eigenvalue derivatives occur, use Eq.(11) to determine the differentiable eigenvectors, use Eq.(33) to determine \( W_1 \). Then, calculate the coefficient matrix \( s_1 \) by Eqs.(34), (35) and (37).
4. If repeated first eigenvalue derivatives occur, solve eigenvalue problem, Eq.(45), to obtain the second eigenvalue derivatives. Determine the differentiable eigenvectors for distinct second eigenvalue derivatives case. Then determine \( W_1 \) by Eq.(33) and compute \( s_1 \) using Eqs.(46) to (52).

For the case in which some eigenvalue derivatives of second order are still repeated, the complete set of unique differentiable eigenvectors cannot be identified yet. Thus, the eigenproblems associated with the eigenvalue derivatives of higher orders have to be solved until one of the higher derivatives of the repeated eigenvalues becomes distinct. This is not very difficult to do since the general forms of the higher derivatives of the eigenvalues and eigenvectors have been presented in this paper, as shown in Eqs.(12) to (21).

### NUMERICAL EXAMPLES

#### Example 1

To check the validity of the present method, the three-order problem used by Juang et al. (1989) is re-examined, but only the eigenvector derivatives associated with the repeated eigenvalues are calculated. The left and right eigenvectors of concern are
\[
X_L = X_R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
Then, using Eq.(28) we get
\[
G_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
Solving the eigenvalue problem of Eq.(31), we obtain
\[
\Lambda^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_L = \Gamma_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
Since the eigenvalue derivatives are also repeated and \( h = 1 \), the differentiable eigenvectors cannot be identified yet. Hence, solve the eigenproblem of Eq.(45) and get
\[
\Lambda^{(2)} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \quad a_1 = b_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\]
The second derivatives of the repeated eigenvalues are distinct. Hence, the differentiable eigenvectors can be determined
\[
Z_R = X_R \Gamma_R a_1 = X_R, \quad Z_L = X_L
\]
Then, from Eq.(33) we have
\[
W_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
Since all the first eigenvalue derivatives are repeated, the off-diagonal elements of \( s_1 \) are directly calculated by using Eqs.(48) to (52). Combining Eq.(37), we get
\[
s_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}
\]
Finally, the derivatives of the eigenvectors under consideration are
\[
Z_R^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
which is the same as given by Juang et al. (1989).

#### Example 2
Assume there exists a five-order problem where

\[
M = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

\[K(p) = \begin{pmatrix}
 k_1 & 0 & 0 & 0 & -k_1 \\
(p-1)^2 & p^2 & 0 & 0 & -p^2 \\
0 & 0 & k_2 & 0 & -k_2 \\
0 & 0 & 0 & k_3 & -k_3 \\
-k_1 & -p -k_2 & -k_3 & -p & -p
\end{pmatrix}
\]

where

\[k_1 = \frac{1}{3}(1 + 2p^3) \quad k_2 = 2p^2 - 2p + 1 \]

\[k_3 = \frac{1}{3}(1 + 2p^3) + 3p^2 - p + 7 \]

The eigenvector derivatives are required at \( p = p_0 = 1 \). First, solving the generalized eigenvalue problem for \( p_0 = 1 \) gives the eigenvalues 0.5, 1, 1, 1, 3 and the normalized left and right eigenvectors associated with the repeated eigenvalues

\[ X_L^T = X_R^T = \frac{1}{2\sqrt{6}} \begin{pmatrix}
2\sqrt{3} & -2\sqrt{3} & 0 & 0 & 0 \\
2 & 2 & -4 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -3\sqrt{2} & 0
\end{pmatrix} \]

Hence, it is obtained

\[
G_s = \begin{pmatrix}
3 & 3 & 3 & 3 & 2 \\
3 & 3 & 3 & 3 & 2 \\
3 & 3 & 3 & 3 & 2 \\
2 & 2 & 2 & 2 & 0
\end{pmatrix}
\]

Solve Eq.(31), we get

\[ \Lambda^{(1)} = \text{diag}(2, 2, 1.25), \quad \Gamma_L = \Gamma_R = I_3 \]

Since repeated first eigenvalue derivatives occur, Eq.(45) is further solved which gives

\[ \Lambda^{(2)} = \text{diag}(1.6, 4), \quad \Lambda^{(3)} = \frac{31}{48} \]

\[ a_1 = \left( \begin{array}{c}
-\sqrt{3/2} \\
1/2 \\
\sqrt{3/2}
\end{array} \right), \quad a_2 = 1 \]

Thus, the differentiable eigenvectors can be identified

\[ Z_L^T = Z_R^T = \frac{1}{2\sqrt{6}} \begin{pmatrix}
-2 & 4 & -2 & 0 & 0 \\
2 & 0 & 0 & -2\sqrt{3} & 0 \\
\sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & -3\sqrt{2} \\
1 & 1 & 1 & 1 & 2
\end{pmatrix} \]

and the particular solution is

\[ W_1 = \sqrt{3} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

The coefficient matrix can be obtained

\[ s_1 = \begin{pmatrix}
0.00000 & 0.57735 & 0.31427 \\
0.28868 & 0.00000 & 0.00000 \\
-0.31427 & 0.00000 & 0.00000
\end{pmatrix} \]

The eigenvector derivatives are

\[ Z_R^{(1)} = \begin{pmatrix}
0.11340 & -23570 & -27665 \\
-0.9072 & 0.47140 & 0.36465 \\
-0.29485 & -23570 & -0.02005 \\
0.27717 & 0.00000 & 0.10825 \\
0.00000 & 0.00000 & 0.21651
\end{pmatrix} \]

**CONCLUSIONS**

This paper has developed a new method for computation of the derivatives of eigenvalue and eigenvector derivatives associated with repeated eigenvalues of the generalized nondefective eigenproblem. This method requires the knowledge of only those eigenvectors to be differentiated. A complete set of formulations has been presented covering the case where some groups of repeated first eigenvalue derivatives occur. In addition, the proposed approach can be extended to deal with general cases in which repeated eigenvalue derivatives of higher, but finite, orders than that discussed in this paper occur. Also, if required, this approach can be used to compute the higher derivatives of the eigenvectors associated with repeated eigenvalues, since the general forms of the higher derivatives of the eigenvalues and eigenvectors have been presented in this paper, as shown in Eqs.(12) to (21). Moreover, it is not difficult to verify that the algorithm presented by Juang et al. (1989) is just a special case of that presented in this paper, since the formulations presented by Juang et al. (1989) for computation of eigenvector derivatives can be easily obtained by substituting Eq.(29) into the formulations presented in this paper.

**REFERENCES**


APPENDIX A

Observe that Eq. (12) is solvable if and only if
\[ \text{rank}(\hat{K} F_i) = \text{rank}(\hat{K}) \] (A1)

Hence, Eq. (16) can be proved as follows.

Proof. Necessity. If Eq. (A1) holds, then \( F_i \) is a linear combination of the column vectors of \( \hat{K} \), i.e.,

\[ F_i = \hat{K} B \] (A2)

Premultiplying both sides of this equation by \( Z_L^T \) with the aid of Eq.(2) gives
\[ Z_L^T F_i = Z_L^T \hat{K} B = 0 \] (A2)

Sufficiency. If \( Z_L^T F_i = 0 \), using \( Z_L^T \hat{K} = 0 \), we get \( Z_L^T [\hat{K} F_i] = 0 \), or \( \hat{K}^T Z_L = 0 \), which means \( \text{dim}(\hat{K}) \geq n \). Hence

\[ \text{rank}(\hat{K}) = \text{rank}(\hat{K}^T) = N - \text{dim}(\hat{K}^T) \leq N - n = r \] (A3)

On the other hand, it is clear that \( \hat{K} \subset \hat{R} \), which implies
\[ \text{rank}(\hat{R}) \geq r = \text{rank}(\hat{K}) \] (A4)

Therefore, a combination of Eqs. (A3) and (A4) leads to
\[ \text{rank}(\hat{R}) = r \] (A5)

This completes the proof.