

Fuzzy numerical solution to the unconfined aquifer problem under the Boussinesq equation

N. Samarinas, C. Tzimopoulos and C. Evangelides

ABSTRACT

In this article, the fuzzy numerical solution of the linearized one dimensional Boussinesq equation of unsteady flow in a semi-infinite unconfined aquifer bordering a lake is examined. The equation describing the problem is a partial differential parabolic equation of second order. This equation requires the knowledge of the initial and boundary conditions as well as the various soil parameters. The above auxiliary conditions are subject to different kinds of uncertainty due to human and machine imprecision and create ambiguities to the solution of the problem and a fuzzy method is introduced. Since the physical problem refers to a partial differential equation, the generalized Hukuhara (gH) derivative was used, as well as the extension of this theory regarding the partial derivatives. The objective of this paper is to compare the fuzzy numerical and analytical results, for two different cases of physical problem of aquifer's unsteady flow, in order to prove the reliability and efficiency of the proposed fuzzy numerical scheme (fuzzy Crank-Nicolson scheme). The comparison of the methods was based on the transformed Hausdorff metric, presented that the distances between the analytical and numerical results tend to zero.

Key words | Boussinesq equation, depth profiles, fuzzy finite difference, fuzzy partial differential equation, gH-derivative

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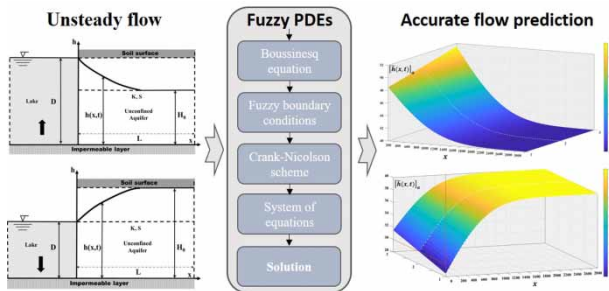
HIGHLIGHTS

- Novel fuzzy numerical scheme.
- Solve the Boussinesq equation in a fuzzy environment.
- Compare the fuzzy scheme with the corresponding analytical results.
- Include the uncertainties of the physical problem.
- Provides a strong advantage to decision makers for efficient water management planning.

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GRAPHICAL ABSTRACT



INTRODUCTION

The Boussinesq equation proposed by (Boussinesq 1904), is one dimensional nonlinear parabolic second order partial differential equation and describes the physical problem of the horizontal water flow concerning unconfined aquifers without precipitation. For this equation the following assumptions apply: (1) the inertial forces are negligible and (2) the horizontal component of velocity V_x does not vary depending on depth and its function of (x, t) . Many researchers have dealt with classical (two-valued) logic and gave analytical (Boussinesq 1904; Polubarinova-Kochina 1949, 2015; Yeh 1970; Glover 1978; Koussis 1979; De Ridder & Zijlstra 1994; Zhang *et al.* 2014) and numerical (Neuman & Witherspoon 1971; van Schilfgaard 1974; Tzimopoulos & Terzidis 1975; Tzimopoulos 1976; Colmenares & Neilan 2016) solutions to the equation but all of the aforementioned problems convey fuzziness (Cox 1994). In general, until recently, in practice, an attempt has been made to solve a classical mechanic problem introducing many uncertainties in the solution process.

The physical problem described by Boussinesq equation can produce ambiguities and uncertainties regarding: the definition of the initial flow condition, the way Boussinesq's equation becomes linearized, the definition of drain spacing and the hydraulic conductivity, boundary conditions, etc (Tzimopoulos *et al.* 2018a, 2018b). The uncertainties and ambiguities that groundwater physical problems present must be taken into account, and as not being included in the final calculations, can lead to incorrect conclusions and management practices, with disastrous consequences for surface and groundwater resources and therefore

enormous environmental, social and economic impacts. For all the above reasons, the solution of this problem should be obtained by fuzzy algorithms.

Fuzzy logic theory is derived from the development of fuzzy sets theory of Lofti Zadeh (Zadeh 1965) and is a powerful tool to model ambiguity and its development gave a big boost not only to theoretical problems (Bhaskar *et al.* 2004; Nieto & Rodríguez-López 2006; Aminikhah 2015) but also to engineering and hydraulic problems (Guo *et al.* 2003b, 2003a). When we are studying in fields of physics and engineering, we often meet problems of fuzzy partial differential equations which have to be solved by numerical methods because the exact solution of these problems, as in the classical logic, can be found only in some special cases (Boussinesq 1904; Polubarinova-Kochina 2015). This case of numerical methods for solving fuzzy differential equations have been rapidly growing in recent years.

During the last years, some analytical and numerical methods were proposed in order to solve fuzzy differential equations. Initially, the concept of fuzzy derivative was introduced by Chang and Zadeh (Chang & Zadeh 1972), followed by Dubois and Prade (Dubois & Prade 1982), who used extension principle in their approach. Fuzzy differential functions were studied by Puri and Ralescu (Puri & Ralescu 1983), who extended Hukuhara's derivative (H-derivative) (Hukuhara 1967) of a set of values appearing in fuzzy sets. Kaleva and Seikkala in (Kaleva 1987, 1990; Seikkala 1987) developed the fuzzy initial value problem. But this method has presented certain drawbacks, and in many cases this solution was not a good generalization of the classic case. The generalized Hukuhara

differentiability (gH – differentiability) was introduced by Bede and Gal (Bede & Gal 2005) and Stefanini and Bede (Stefanini & Bede 2009) which overcomes this drawback. This new derivative is defined for a larger class of fuzzy functions than Hukuhara derivative. Allahviranloo *et al.* in (Allahviranloo *et al.* 2015), introduced the (gH-p) differentiability for partial derivatives as an extension of the above theory. Tzimopoulos *et al.* in (Tzimopoulos *et al.* 2018a, 2018b) used the above method and gave a fuzzy analytical solution to a parabolic differential equation and also Tzimopoulos *et al.* in (Tzimopoulos *et al.* 2018a, 2018b, 2020) gave a fuzzy analytical solution to a unconfined aquifer problem described by Boussinesq equation.

Analytical methods approach real problems with changing parameters in space and time only in limiting cases and this disadvantage leads to the use of numerical methods such as finite difference methods transformed in fuzzy environment, which approach any physical problem. The numerical method for solving fuzzy differential equations was introduced by Ma *et al.* (Ma *et al.* 1999). Subsequently, numerical solutions of fuzzy differential equations were examined by Friedman (Friedman *et al.* 1999), Bede (Bede 2006) and Abbasbandy (Abbasbandy & Allahviranloo 2002). The existence of solution for fuzzy partial differential equations were investigated also by Buckley and Feuring in (Buckley & Feuring 1999); their proposed method works only for elementary partial differential equations. Based on Seikkala derivative Allahviranloo *et al.* in (Allahviranloo 2002), and Kermani *et al.* in (Kermani & Saburi 2007) use a numerical method which is an explicit difference method to solve partial differential equations. Farajzadeh in (Farajzadeh 2010) gives an explicit method for solving fuzzy partial differential equation. Uthirasamy in (Uthirasamy 2014) gives studies on numerical solutions of fuzzy boundary value problems and fuzzy partial differential equations. More recently, Samarinas *et al.* in (Samarinas *et al.* 2018) gives a fuzzy implicit scheme to solve numerically the physical problem of the horizontal infiltration.

In this article, a fuzzy numerical solution of the linearized one dimensional Boussinesq equation, with initial and boundary conditions, is presented applying the Crank-Nicolson scheme (Crank & Nicolson 1947) in fuzzy environment. Two different cases of the problem of unsteady flow in a semi-infinite unconfined aquifer bordering a lake are examined. In the first case there is a sudden rise of the lake's

water level, thus the aquifer recharging from the lake and for the second case a sudden drop of the lake's water level, thus the aquifer discharging to the lake. In both cases special emphasis is given to the aquifer boundary conditions, which are considered to be uncertain, as their uncertainties are the ones that affect to a greater extent the solution of the problem. Moreover, the hydraulic parameters of this problem are considered crisp as well as the geometric parameters. In general the hydraulic communication between the lake and the aquifer has an important effect on the control of riparian ecosystem and it can also alter the water chemistry. The engineers must have the knowledge of the above effects on ecological and hydrological processes for water resource management purposes.

MATERIALS AND METHODS

In order to present in a comprehensive way the methodology used in this work, initially for the understanding of the physical problem and the difficulties it presents but also for the development of its solution, it is considered appropriate to present the following flowchart (Figure 1). In this flowchart, the methodology of solving the physical problem is developed, determining that the ambiguities of the problem require the use of fuzzy partial differential equation for accurate results. Furthermore, emphasize is given to the two different approaches of solving fuzzy partial differential equations and highlight the advantages and disadvantages of each method in order to solve the problem.

Physical problem

In the cases of the aquifer illustrated in Figure 2(a) and 2(b) a rise and drop in the lake's water level is observed and the aquifer flow is described by the following Boussinesq equation:

$$\frac{\partial h}{\partial t} = \frac{K}{S} \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right)$$

or by its linear representation (De Ridder & Zijlstra 1994):

$$\frac{\partial h}{\partial t} = \frac{KD}{S} \frac{\partial^2 h}{\partial x^2} = a \frac{\partial^2 h}{\partial x^2}, \quad \text{with } a = \frac{KD}{S}$$

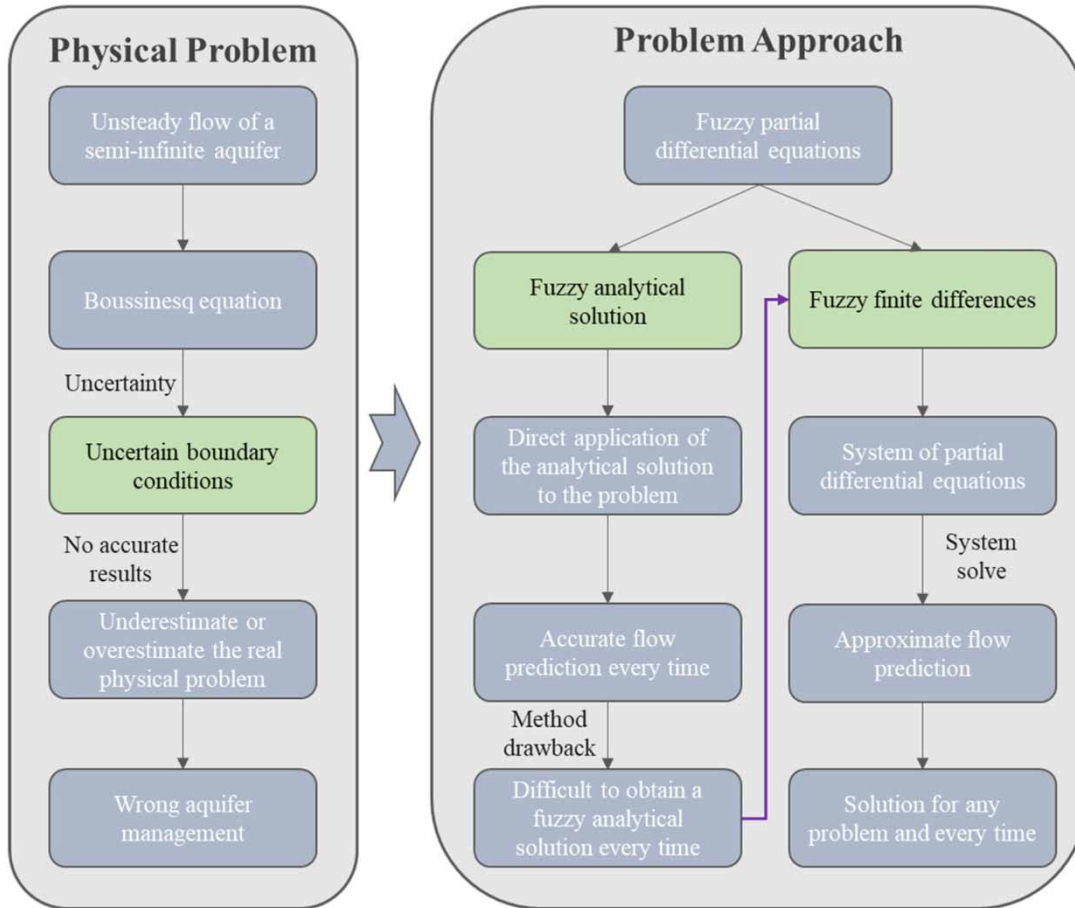


Figure 1 | Methodology flowchart.

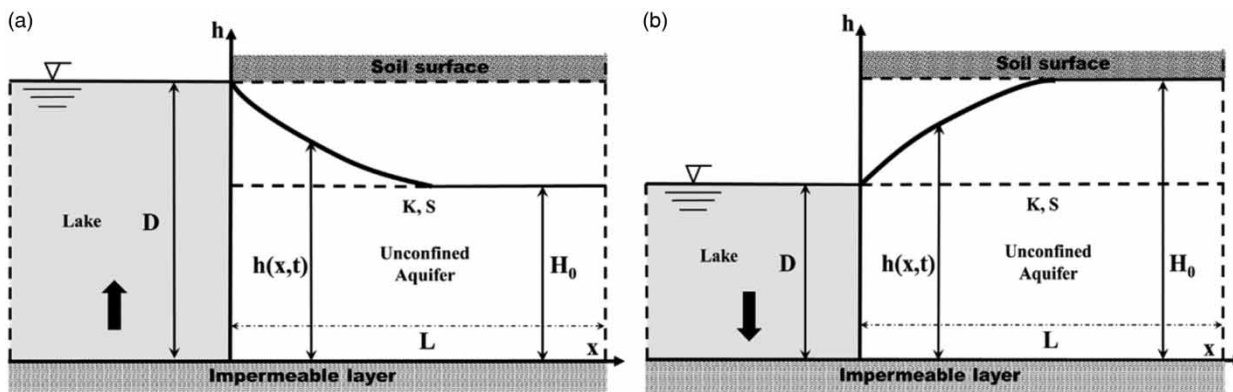


Figure 2 | (a) Aquifer recharging from the lake, (b) Aquifer discharging to the lake.

where \mathbf{K} = the hydraulic conductivity of the aquifer, \mathbf{S} = the specific yield of the aquifer or drainable pore space, \mathbf{D} = the depth of the lake, $\mathbf{h}(\mathbf{x}, \mathbf{t})$ = the depth of the aquifer and \mathbf{x} , \mathbf{t} = the coordinates (spatial and temporal).

The boundary conditions of the problem are:

$$t > 0, \quad h(0, t) = D, \quad h(x, t) = H_0 \quad x \rightarrow \infty$$

while initial condition is:

$$t = 0, \quad h(x, 0) = H_0$$

To facilitate calculations non-dimensional variables are introduced:

$$H = \frac{h - H_0}{D - H_0}, \quad \xi = \frac{x}{L}, \quad \tau = \frac{KDt}{SL^2} = \frac{at}{L^2}$$

where L = a specific length. The new resulting equation is as follows:

$$\frac{\partial H}{\partial \tau} = \frac{\partial^2 H}{\partial \xi^2} \tag{1}$$

with the new boundary and initial conditions:

$$\tau > 0, \quad H(0, \tau) = 1, \quad H(\xi, \tau) = 0, \quad \tau = 0, \quad H(\xi, 0) = 0$$

Fuzzy Sets and Hukuhara generalized partial derivative

Definition 1. We denote by \mathbb{R}_F the class of fuzzy subsets $\tilde{u}: \mathbb{R}_f \rightarrow [0, 1]$, satisfying the following properties (Puri & Ralescu 1985; Kaleva 1987):

1. \tilde{u} is normal, that is, there exists, $x_0 \in \mathbb{R}$ with $u(x_0) = 1$
2. \tilde{u} is a convex fuzzy set. that is, $\tilde{u}(\lambda x + (1 - \lambda)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\}, \forall x, y \in \mathbb{R}, \forall \lambda \in [0, 1]$
3. \tilde{u} is upper semi-continuous on \mathbb{R} .
4. $\overline{\{x \in \mathbb{R} | u(x) > 0\}}$ is compact, where \bar{A} denotes the closure of A

Then \mathbb{R}_F is called the space of fuzzy numbers.

Definition 2. Let \tilde{u} be a fuzzy number. Fundamental concepts in fuzzy theory are the support, the level-sets (or level-cuts) and the core of a fuzzy number. The a -level set of $[\tilde{u}]_a$ (or simply a -cut) is defined by $[\tilde{u}]_a = \{x | x \in \mathbb{R}, \tilde{u}(x) \geq a\}$ and for $a = 0$ it is the closure of the support $[\tilde{u}]_0 = cl\{x \in \mathbb{R}, u(x) > 0\}$ (Stefanini & Bede 2014).

Definition 3. The necessary and sufficient conditions for $[\tilde{u}]_a = (u_a^-, u_a^+)$ to define a fuzzy number are as follows (Goetschel & Voxman 1986; Stefanini et al. 2006; Nieto et al. 2009):

1. u_a^- is a bounded monotonic non-decreasing left-continuous function for all $a \in [0, 1]$ and right-continuous for $a = 0$.
2. u_a^+ is a bounded monotonic non-increasing left-continuous function for all $a \in [0, 1]$ and right-continuous for $a = 0$.
3. $u_1^- \leq u_1^+$, for $a = 1$, which implies $u_a^- \leq u_a^+ \forall a \in [0, 1]$

Definition 4. The metric structure is given by the transformed Hausdorff distance, in fuzzy sets, $D: \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}_+ \cup \{0\}$, by (Lakshimikantham et al. 2006; Nieto et al. 2009):

$$D(u, v) = \sup_{a \in [0,1]} \max\{|u_a^- - v_a^-|, |u_a^+ - v_a^+|\} \tag{2}$$

where $[\tilde{u}]_a = [u_a^-, u_a^+]$, $[\tilde{v}]_a = [v_a^-, v_a^+]$

Then it is easy to see that D is a metric in \mathbb{R}_F and has the following properties:

1. $D(u + w, v + w) = D(u, v)$
2. $D(\lambda u, \lambda v) = |\lambda|D(u, v)$
3. $D(u + v, w + e) \leq D(u, w) + D(v, e)$ and (\mathbb{R}_F, D) is a complete metric space, for all $u, v, w, e \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$.

Definition 5. The generalized Hukuhara difference of two fuzzy numbers $\tilde{u}, \tilde{v} \in \mathbb{R}_F$ is defined as follows (Bede & Stefanini 2013):

$$\tilde{u} \ominus_{gH} \tilde{v} = w \Leftrightarrow \begin{cases} (i) & u = v + w \\ or (ii) & v = u + (-1w) \end{cases}$$

In terms of a -levels we have:

$$[\tilde{u} \ominus_{gH} \tilde{v}]_a = [\tilde{w}]_a = [w_a^-, w_a^+] = [\min\{u_a^- - v_a^-, u_a^+ - v_a^+\}, \max\{u_a^- - v_a^-, u_a^+ - v_a^+\}]$$

and if the H-difference exists, then $\tilde{u} \ominus_H \tilde{v} = \tilde{u} \ominus_{gH} \tilde{v}$; the condition of existence of $\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w}$ are w_a^- non decreasing and w_a^+ non increasing with $w_a^- < w_a^+$.

Definition 6.

A. First order

A fuzzy-valued function H of two variables is a rule that assigns to each ordered pair of real numbers (x, t) , in a set D a unique fuzzy number denoted by $\tilde{H}(x, t)$. Let $\tilde{H}(x, t): D \rightarrow \mathbb{R}_F, (x_0, t_0) \in D$ and $H_a^-(x, t), H_a^+(x, t)$ be real valued functions and partial differentiable with respect to x and t . We say that (Khastan & Nieto 2010; Allahviranloo et al. 2015; Mondal & Roy 2015):

- $\tilde{H}(x, t)$ is [(i)-p]-differentiable w.r.t x at (x_0, t_0) if

$$\frac{\partial \tilde{H}_a(x_0, t_0)}{\partial x_{i, gH}} = \left[\frac{\partial H_a^-(x_0, t_0)}{\partial x}, \frac{\partial H_a^+(x_0, t_0)}{\partial x} \right]$$

- $\tilde{H}(x, t)$ is [(ii)-p]-differentiable w.r.t x at (x_0, t_0) if

$$\frac{\partial \tilde{H}_a(x_0, t_0)}{\partial x_{i, gH}} = \left[\frac{\partial H_a^+(x_0, t_0)}{\partial x}, \frac{\partial H_a^-(x_0, t_0)}{\partial x} \right]$$

Notation. The same is valid for $\frac{\partial \tilde{H}_a(x_0, t_0)}{\partial t}$

Definition 7.

B. Second order

Let $\tilde{H}(x, t): D \rightarrow \mathbb{R}_F$, and $\partial \tilde{H}_a(x_0, t_0) / \partial x_{i, gH}$ be [gH-p]-differentiable at $(x_0, t_0) \in D$ with respect to x . We say that (Khastan & Nieto 2010; Allahviranloo et al. 2015):

1. $\frac{\partial \tilde{H}_a(x_0, t_0)}{\partial x_{i, gH}}$ is [(i)-p]- differentiable w.r.t x if:

$$\frac{\partial^2 \tilde{H}_a(x_0, t_0)}{\partial x_{i, gH}^2} = \begin{cases} \frac{\partial^2 H_a^-(x_0, t_0)}{\partial x^2}, \frac{\partial^2 H_a^+(x_0, t_0)}{\partial x^2} \text{ if } \tilde{H}(x, t) \text{ is [(i) - p]differentiable} \\ \frac{\partial^2 H_a^+(x_0, t_0)}{\partial x^2}, \frac{\partial^2 H_a^-(x_0, t_0)}{\partial x^2} \text{ if } \tilde{H}(x, t) \text{ is [(ii) - p]differentiable} \end{cases}$$

2. $\frac{\partial \tilde{H}_a(x_0, t_0)}{\partial x_{ii, gH}}$ is [(ii)-p]- differentiable w.r.t x if:

$$\frac{\partial^2 \tilde{H}_a(x_0, t_0)}{\partial x_{ii, gH}^2} = \begin{cases} \frac{\partial^2 H_a^+(x_0, t_0)}{\partial x^2}, \frac{\partial^2 H_a^-(x_0, t_0)}{\partial x^2} \text{ if } \tilde{H}(x, t) \text{ is [(i) - p]differentiable} \\ \frac{\partial^2 H_a^-(x_0, t_0)}{\partial x^2}, \frac{\partial^2 H_a^+(x_0, t_0)}{\partial x^2} \text{ if } \tilde{H}(x, t) \text{ is [(ii) - p]differentiable} \end{cases}$$

Transform of the fuzzy problem

System of Crisp Problems

The problem is transformed now with the help of the generalized Hukuhara difference and Hukurara partial derivative

to the following fuzzy partial differential equation:

$$\frac{\partial \tilde{H}}{\partial \tau} \ominus_{gH} \frac{\partial^2 \tilde{H}}{\partial \xi^2} = 0 \tag{3}$$

with the new boundary and initial conditions:

$$\begin{aligned} \left[\tilde{H}(\xi, \tau) \right]_{\xi=0}^{\alpha} &= [1 - c(1 - \alpha), 1 + c(1 - \alpha)], \\ \tau > 0, \left[\tilde{H}(\xi, \tau) \right]_{\xi=\infty}^{\alpha} &= \{\tilde{0}\}, \left[\tilde{H}(\xi, \tau) \right]_{\tau=0}^{\alpha} = \{\tilde{0}\} \end{aligned} \tag{4}$$

We can find solutions to the fuzzy Equation (3) with the boundary and initial conditions (4) according to (Tzimopoulos et al. 2018a, 2018b) utilizing the theory of (Khastan & Nieto 2010; Bede & Stefanini 2011; Bede & Stefanini 2013; Allahviranloo et al. 2015), by translating the above fuzzy problem to a system of second order of crisp boundary value problems (BVPs), hereafter called corresponding system of the fuzzy problem (Khastan & Nieto 2010). Therefore, four BVPs systems are possible for the fuzzy problem, as follows:

System (1,1):

$$\begin{aligned} \frac{\partial H_a^-}{\partial \tau} - \frac{\partial^2 H_a^-}{\partial \xi^2} &= 0 \\ \frac{\partial H_a^+}{\partial \tau} - \frac{\partial^2 H_a^+}{\partial \xi^2} &= 0 \end{aligned}$$

System (1,2):

$$\begin{aligned} \frac{\partial H_a^-}{\partial \tau} - \frac{\partial^2 H_a^+}{\partial \xi^2} &= 0 \\ \frac{\partial H_a^+}{\partial \tau} - \frac{\partial^2 H_a^-}{\partial \xi^2} &= 0 \end{aligned}$$

System (2,1):

$$\begin{aligned} \frac{\partial H_a^+}{\partial \tau} - \frac{\partial^2 H_a^-}{\partial \xi^2} &= 0 \\ \frac{\partial H_a^-}{\partial \tau} - \frac{\partial^2 H_a^+}{\partial \xi^2} &= 0 \end{aligned}$$

System (2,2):

$$\begin{aligned} \frac{\partial H_a^+}{\partial \tau} - \frac{\partial^2 H_a^+}{\partial \xi^2} &= 0 \\ \frac{\partial H_a^-}{\partial \tau} - \frac{\partial^2 H_a^-}{\partial \xi^2} &= 0 \end{aligned}$$

where for all the above systems defined (Figure 3):

$$H_a^-(0, \tau) = 1 - c(1 - \alpha), H_a^-(\infty, \tau) = 0, H_a^-(\xi, 0) = 0$$

$$H_a^+(0, \tau) = 1 + c(1 - \alpha), H_a^+(\infty, \tau) = 0, H_a^+(\xi, 0) = 0$$

The method of solving the above problem is based on the selection of appropriate type of derivative, according to the theory of (Khastan & Nieto 2010; Bede & Stefanini 2013).

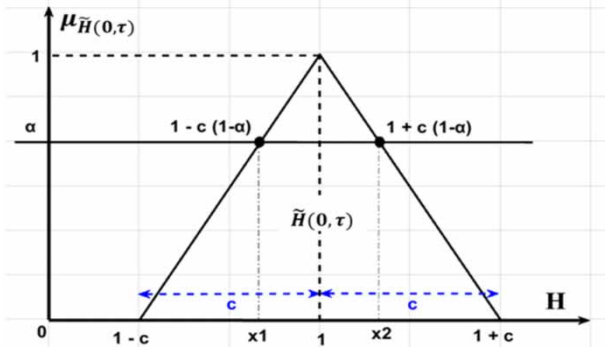


Figure 3 | Fuzzy boundary condition $\tilde{H}(0, \tau)$.

With such a selection, the above problem is transformed to a corresponding system of boundary value problems.

A domain with the valid solution results, in which the derivatives have level sets according to the above theory of differentiability. Analytical solutions of the above systems can be found in (Tzimopoulos et al. 2018a, 2018b).

Fuzzy finite difference

The main idea in the finite difference method is to replace the derivative in a partial differential equation with difference quotients.

Assume that u is a function of the independent crisp variables x and t . Subdivide the x - t plane into sets of equal rectangles of sides h, k , by equally spaced grid lines parallel to O_x , defined by $x_i = ih, i = 0, 1, 2, \dots, m$, and equally spaced grid lines parallel to O_t , defined by $t_j = jk, j = 0, 1, 2, \dots, n$, where m and n are positive integers with $h = L/m$ and $k = T/n$ (Figure 3).

For the development of the fuzzy finite difference scheme we will recall the fuzzy linear partial differential Equation (3):

$$\frac{\partial \tilde{H}}{\partial \tau} \ominus_{gH} \frac{\partial^2 \tilde{H}}{\partial \xi^2} = 0 \tag{5}$$

with the non-dimensional boundary and initial conditions:

$$\left[\tilde{H}(\xi, \tau) \right]_{\alpha} = [1 - c(1 - \alpha), 1 + c(1 - \alpha)]$$

$$t > 0, \left[H(x, t) \right]_{\alpha} = \{\tilde{0}\}, \left[\tilde{H}(\xi, \tau) \right]_{\alpha} = \{\tilde{0}\}$$

If **System 1** satisfies the Equation (5) then the fuzzy Crank-Nicolson scheme develops as follows.

By using implicit discretization scheme and applying Taylor's theorem and definition of standard difference we get the approximate formulae:

$$\frac{H_{i,j+1}^- - H_{i,j}^-}{\Delta \tau} = \frac{1}{2} \cdot \left\{ \frac{H_{i+1,j+1}^- - 2H_{i,j+1}^- + H_{i-1,j+1}^-}{(\Delta \xi)^2} + \frac{H_{i+1,j}^- - 2H_{i,j}^- + H_{i-1,j}^-}{(\Delta \xi)^2} \right\} + O(\Delta \tau) + O(\Delta \xi)^2 \tag{6a}$$

$$\frac{H_{i,j+1}^+ - H_{i,j}^+}{\Delta \tau} = \frac{1}{2} \cdot \left\{ \frac{H_{i+1,j+1}^+ - 2H_{i,j+1}^+ + H_{i-1,j+1}^+}{(\Delta x)^2} + \frac{H_{i+1,j}^+ - 2H_{i,j}^+ + H_{i-1,j}^+}{(\Delta x)^2} \right\} + O(\Delta \tau) + O(\Delta \xi)^2 \tag{6b}$$

By using the Crank-Nicolson scheme the combination of (6a) and (6b) gives the following:

$$H_{i,j+1}^- - H_{i,j}^- = r [H_{i+1,j}^- - 2H_{i,j}^- + H_{i-1,j}^- + H_{i+1,j+1}^- - 2H_{i,j+1}^- + H_{i-1,j+1}^-]$$

$$H_{i,j+1}^+ - H_{i,j}^+ = r [H_{i+1,j}^+ - 2H_{i,j}^+ + H_{i-1,j}^+ + H_{i+1,j+1}^+ - 2H_{i,j+1}^+ + H_{i-1,j+1}^+]$$

or the following equations must be hold:

$$-r H_{i-1,j+1}^- + (2 + 2r)H_{i,j+1}^- - r H_{i+1,j+1}^- = rH_{i-1,j}^- + (2 - 2r) H_{i,j}^- + r H_{i+1,j}^- \tag{7a}$$

$$-r H_{i-1,j+1}^+ + (2 + 2r)H_{i,j+1}^+ - r H_{i+1,j+1}^+ = rH_{i-1,j}^+ + (2 - 2r) H_{i,j}^+ + r H_{i+1,j}^+ \tag{7b}$$

where $r = \frac{\Delta \tau}{(\Delta \xi)^2}$

Matrix form of the difference schemes (7a) and (7b) are as follows:

$$P_1 H_{j+1}^- = Q_1 H_j^- + G_{j+1}, \quad P_1 H_{j+1}^+ = Q_1 H_j^+ + G_{j+1}$$

$$P = \begin{bmatrix} 2+2r & -r & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -r & 2+2r & -r & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & -r & 2+2r & -r & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & -r & 2+2r & \dots & \dots & \dots \end{bmatrix},$$

$$Q = \begin{bmatrix} 2-2r & r & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ r & 2-2r & r & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & r & 2-2r & r & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & r & 2-2r & \dots \end{bmatrix}$$

$$H_{j+1}^- = [H_{1,j+1}^-, \dots, H_{N-1,j+1}^-]^T, \quad H_{j+1}^+ = [H_{1,j+1}^+, \dots, H_{N-1,j+1}^+]^T$$

$$G_{j+1} = [r\omega_{0j}, 0, \dots, 0, r\omega_{Nj}]^T$$

where: $\omega_j \in \{H_j^-, H_j^+\}$

If **System 2** satisfies the Equation (5) then the fuzzy Crank-Nicolson scheme develops as follows.

By using implicit discretization scheme and applying Taylor's theorem and definition of standard difference we get the approximate formulae:

$$\frac{H_{ij+1}^- - H_{ij}^-}{\Delta\tau} = \frac{1}{2} \cdot \left\{ \frac{H_{i+1,j+1}^+ - 2H_{ij+1}^+ + H_{i-1,j+1}^+}{(\Delta\xi)^2} + \frac{H_{i+1,j}^+ - 2H_{ij}^+ + H_{i-1,j}^+}{(\Delta\xi)^2} \right\} + O(\Delta\tau) + O(\Delta\xi)^2 \quad (8a)$$

$$\frac{H_{ij+1}^+ - H_{ij}^+}{\Delta t} = \frac{1}{2} \cdot \left\{ \frac{H_{i+1,j+1}^- - 2H_{ij+1}^- + H_{i-1,j+1}^-}{(\Delta x)^2} + \frac{H_{i+1,j}^- - 2H_{ij}^- + H_{i-1,j}^-}{(\Delta x)^2} \right\} + O(\Delta\tau) + O(\Delta\xi)^2 \quad (8b)$$

In this case by using the Crank-Nicolson scheme the combination of (6a), (6b) and (6c) gives the following:

$$H_{ij+1}^- - H_{ij}^- = r[H_{i+1,j}^+ - 2H_{ij}^+ + H_{i-1,j}^+ + H_{i+1,j+1}^+ - 2H_{ij+1}^+ + H_{i-1,j+1}^+]$$

$$H_{ij+1}^+ - H_{ij}^+ = r[H_{i+1,j}^- - 2H_{ij}^- + H_{i-1,j}^- + H_{i+1,j+1}^- - 2H_{ij+1}^- + H_{i-1,j+1}^-]$$

or the following equations must be hold:

$$-rH_{i+1,j+1}^+ + H_{ij+1}^- + 2rH_{ij+1}^+ - rH_{i-1,j+1}^+ = rH_{i-1,j}^+ + H_{ij}^- - 2rH_{ij}^+ + rH_{i+1,j}^+ \quad (9a)$$

$$-rH_{i+1,j+1}^- + H_{ij+1}^+ + 2rH_{ij+1}^- - rH_{i-1,j+1}^- = rH_{i-1,j}^- + H_{ij}^+ - 2rH_{ij}^- + rH_{i+1,j}^- \quad (9b)$$

Matrix form of the difference schemes (9a) and (9b) are as follows:

$$T \begin{bmatrix} H_{j+1}^- \\ H_{j+1}^+ \end{bmatrix} = K \begin{bmatrix} H_j^- \\ H_j^+ \end{bmatrix} + C_{j+1}$$

where

$$T = \begin{pmatrix} R & -Y \\ -Y & R \end{pmatrix}, \quad K = \begin{pmatrix} R & Y \\ Y & R \end{pmatrix}$$

and

$$R = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & \dots & \dots \end{bmatrix}, \quad Y = \begin{bmatrix} -2r & r & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ r & -2r & r & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & r & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & r & -2r & r \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & r & -2r & \dots \end{bmatrix}$$

$$H_{j+1}^- = [H_{1,j+1}^-, \dots, H_{N-1,j+1}^-]^T, \quad H_{j+1}^+ = [H_{1,j+1}^+, \dots, H_{N-1,j+1}^+]^T$$

$$C_{j+1} = [rH_{0j}^-, 0, \dots, 0, rH_{Nj}^-, rH_{0j}^+, 0, \dots, 0, rH_{Nj}^+]^T$$

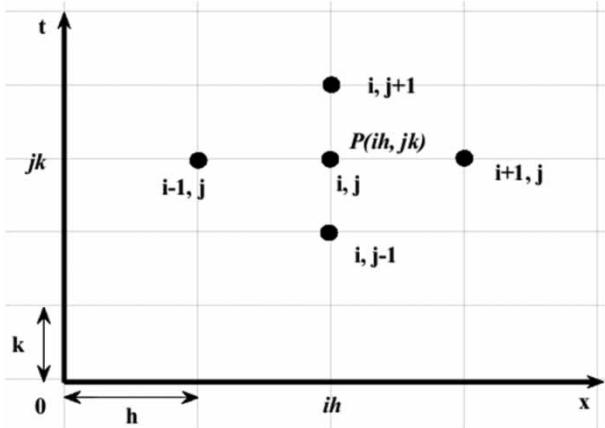


Figure 4 | Grid (Farajzadeh 2010).

In the cases of **Systems (2,1)** and **(2,2)** it can be apparent that the fuzzy Crank-Nicolson formulas are similar to **Systems (1,2)** and **(1,1)** respectively.

RESULTS

For the aquifer of **Figure 2** the following are assumed: $K = 0.01$ cm/sec, $S = 0.15$, $L = 500$ m, $D = 50$ m(recharging case), 30 m(discharging case.), $H_0 = 40$ m and $L = 4,000$ m. According to (Tzimopoulos et al. 2018a, 2018b) the **System 1** considered here, since it gives a physical solution to the problem in both recharging and discharging cases. In both cases fuzziness is introduced on $H(0, \tau)(c = 0.15)$, that is (**Figure 5**):

$$\tilde{H}(0, \tau) = [0.85 + 0.15\alpha, 1.15 - 0.15\alpha]$$

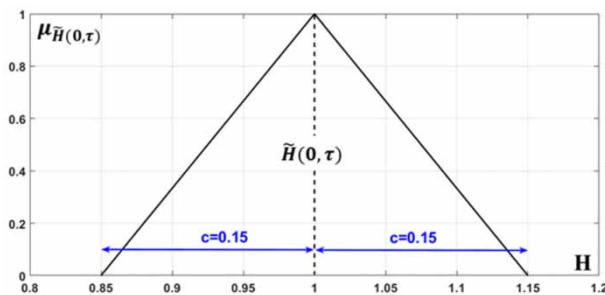


Figure 5 | Fuzzy boundary condition $\tilde{H}(0, \tau)$.

Then the analytical solution becomes (Tzimopoulos et al. 2018a, 2018b):

$$\tilde{H}(x, t) = \left[(0.85 + 0.15\alpha) \cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right), (1.15 - 0.15\alpha) \cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right) \right]$$

Also, the following obtained:

$$h = H_0 + H(D - H_0) \text{ with } D > H_0, \quad D - H_0 > 0, \quad h > H_0$$

– recharging case

$$h = H_0 + H(D - H_0) \text{ with } D < H_0, \quad D - H_0 < 0, \quad h < H_0$$

– discharging case

System 1 satisfies the recharging case and then:

$$[\tilde{H}(0, t)] = [0.85 + 0.15\alpha, 1.15 - 0.15\alpha]$$

$$\begin{aligned} [\tilde{h}(x, t)]_\alpha &= [h_\alpha^-(x, t), h_\alpha^+(x, t)] \\ &= [H_0 + H_\alpha^-(x, t) \cdot (D - H_0), H_0 + H_\alpha^+(x, t) \cdot (D - H_0)] \end{aligned} \tag{10a}$$

where

$$\begin{aligned} H_\alpha^-(x, t) &= (0.85 + 0.15\alpha) \cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right), \\ H_\alpha^+(x, t) &= (1.15 - 0.15\alpha) \cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right) \end{aligned}$$

After using the property of subtraction between fuzzy numbers and the property of fuzzy singleton (Hanss 2005) for the parameter H_0 then **System 1** also satisfies the discharging case with the following form regarding the

analytical solution:

$$[\tilde{h}(x, t)]_\alpha = \left[H_0 - (1.15 - 0.15\alpha) \cdot \operatorname{erfc} \left(\frac{x}{\sqrt{\frac{4KDt}{SL^2}}} \right), H_0 - (0.85 + 0.15\alpha) \cdot \operatorname{erfc} \left(\frac{x}{\sqrt{\frac{4KDt}{SL^2}}} \right) \right] \quad (10b)$$

where

$$\Psi = |D - H_0|$$

Thus, the fuzzy Crank-Nicolson scheme for both cases is:

$$\begin{aligned} -r H_{i-1,j+1}^- + (2 + 2r)H_{ij+1}^- - r H_{i+1,j+1}^- \\ = rH_{i-1,j}^- + (2 - 2r) H_{ij}^- + r H_{i+1,j}^- \\ -r H_{i-1,j+1}^+ + (2 + 2r)H_{ij+1}^+ - r H_{i+1,j+1}^+ \\ = rH_{i-1,j}^+ + (2 - 2r) H_{ij}^+ + r H_{i+1,j}^+ \end{aligned} \quad (11)$$

where $r = dt/(dx)^2$

Generally, when numerical methods are applied, comparison with analytical solution of the problem-when it exists- is a way of verifying the results. In this physical problem the analytical solution exists and for this reason, the fuzzy Crank-Nicolson scheme applied below and compared with the results of the analytical solution. Also, we should mention that the fuzzy Crank-Nicolson scheme solved with the Thomas method(algorithm) which is a simplified form of Gaussian elimination that can be used to solve tri-diagonal system of equations (Smith 1965). Another very important point to note when using differential equations it is that it is a well-posed problem according to Jacques Hadamard (Hadamard 1902).

We used the fuzzy Crank-Nicolson scheme (11) to approximate the fuzzy analytical solutions (10a) and (10b) for different times and points. The optimal spatial and temporal steps that emerged were $dt = 0.0000018$, $dx = 0.01$ thus $r = 0.018$ for the recharging case and $dt = 0.00000108$, $dx = 0.01$ thus $r = 0.0108$ for the discharging case. Figures 6 and 7 illustrates the dimensional depth profiles as a function of x for the recharging and discharging cases respectively.

Figure 8 illustrates, as an example, the membership functions for recharging case for one point in three different times. According to the membership functions, it appears

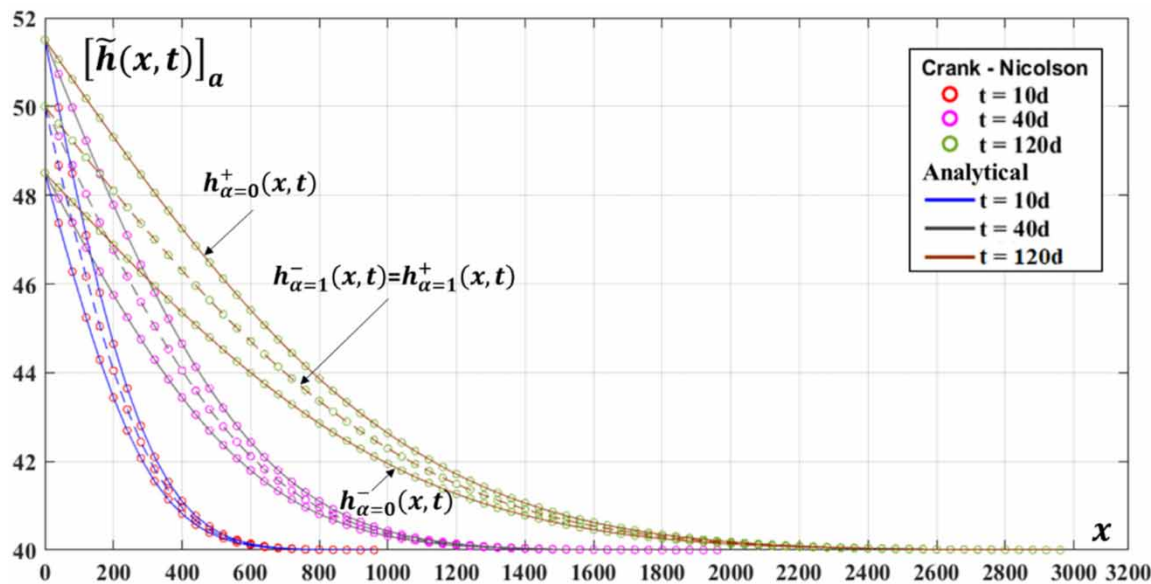


Figure 6 | Dimensional depth profiles for recharging case as a function of x , for the times $t = 10d, 40d$ and $120d$ and for $\alpha = 0$ and 1 .

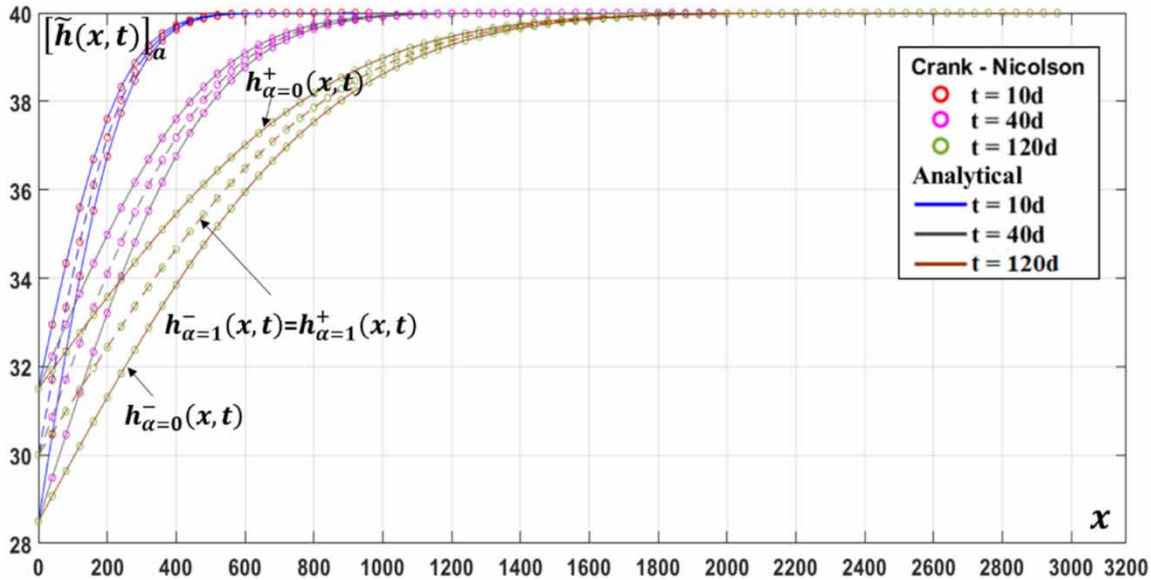


Figure 7 | Dimensional depth profiles for discharging case as a function of x , for the times $t = 10d, 40d$ and $120d$ and for $\alpha = 0$ and 1 .

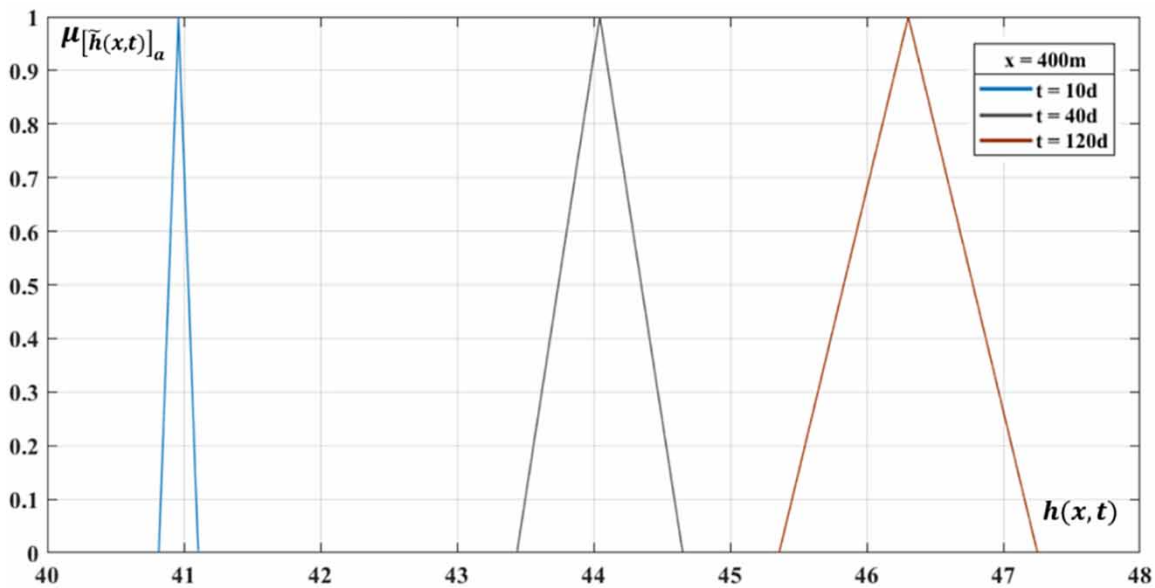


Figure 8 | Membership function $\mu_{[h(x,t)]_a}$ for recharging case for $t = 10d, 40d$ and $120d$ and for $a = 0$.

that the fuzziness follows the time. This means, that as the time tends to grow, higher levels of fuzziness arise.

In addition, Figures 9 and 10 illustrate in 3D mode the dimensional depth profiles as a function of x , using the fuzzy numerical solution, for the time of 120 days, presenting in detail the water movement.

As a first point of view according to above graphs, we can observe that the fuzzy numerical results tend to approach the fuzzy analytical solution satisfactorily. However, a more reliable comparison method, which comes to confirm the visual results, is based on the transformed Hausdorff distance. This metric (Equation (2)) was applied

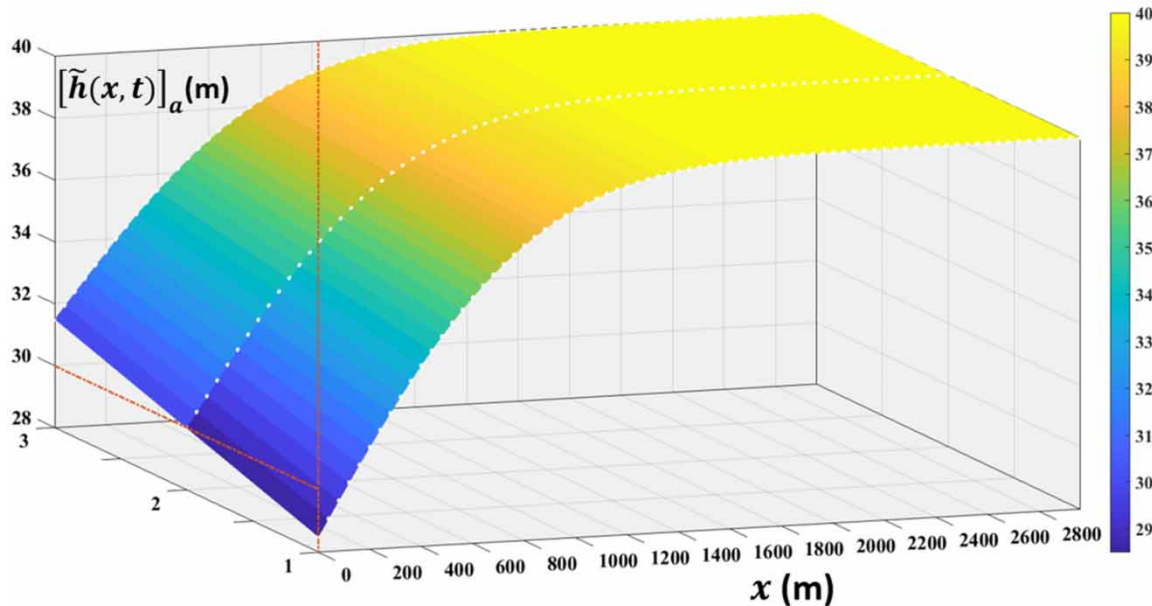


Figure 9 | Dimensional depth profile for discharging case, for $t = 120d$ in 3D mode.

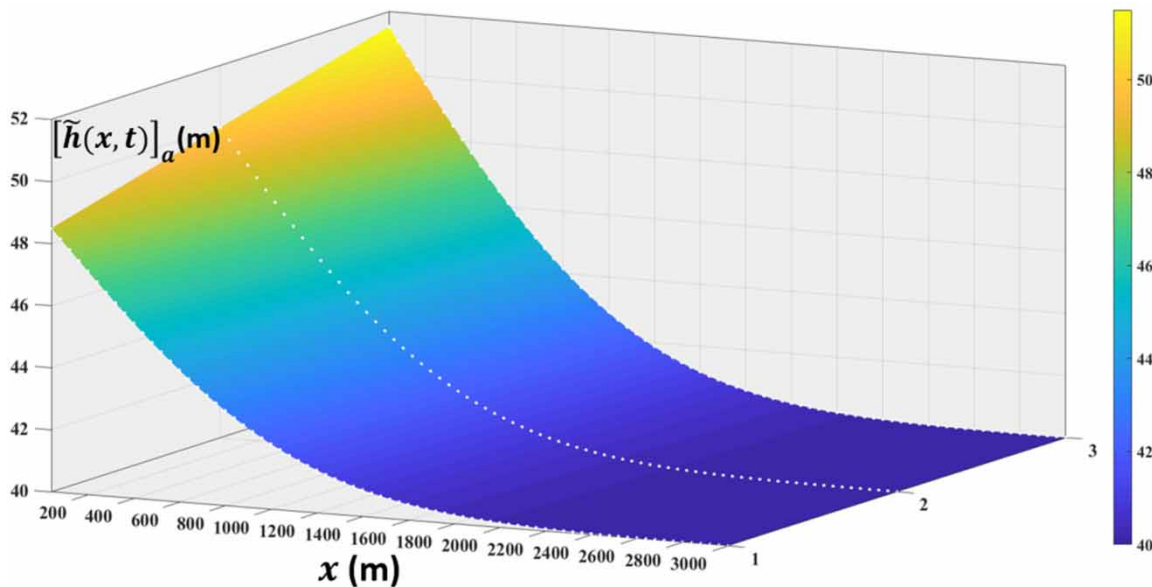


Figure 10 | Dimensional depth profile for recharging case, for $t = 120d$ in 3D mode.

to compare the final results and Figure 11 present the transformed Hausdorff distances for both cases in three different times.

Based on the results observed that the maximum distances between the fuzzy numbers of the two methods

appeared in the results for level $\alpha\text{-cut} = 0$ which is absolutely reasonable since in this $\alpha\text{-cut}$ level the results are extracted taking into account the full percentage of fuzziness. In addition, it was observed that the distances of the fuzzy numbers decrease as the observation times

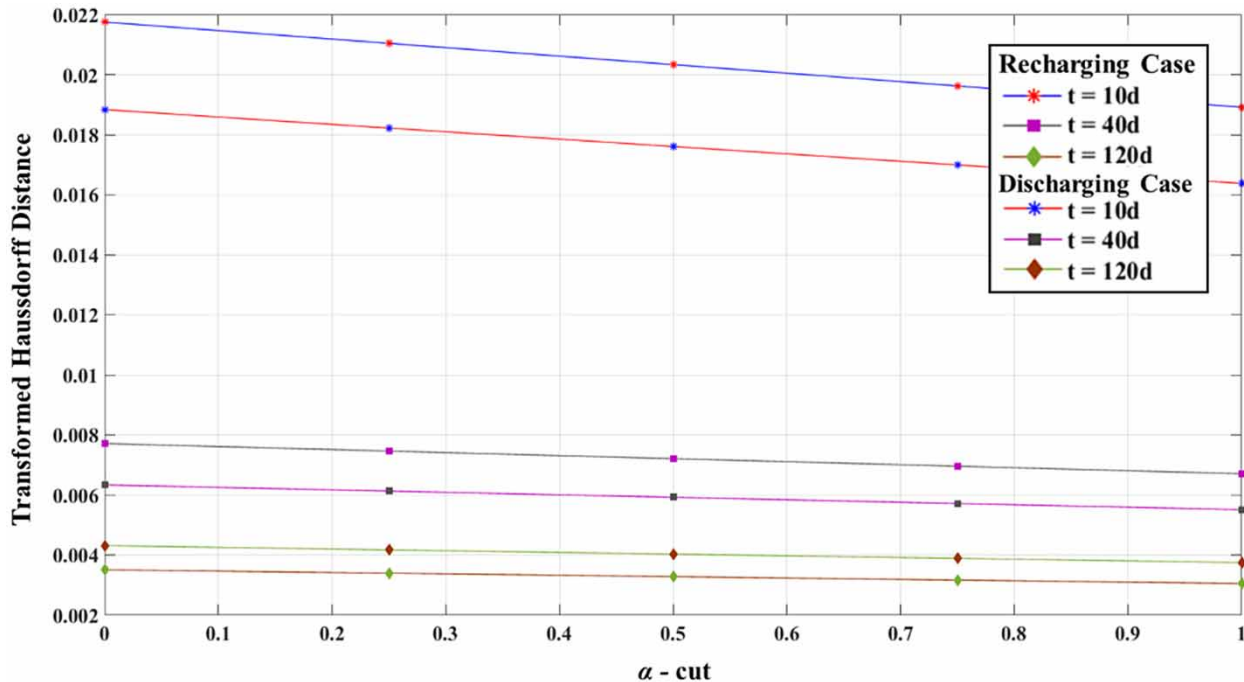


Figure 11 | Transformed Hausdorff distance for $\alpha = 0, 0.25, 0.5, 0.75, 1$ and for the times $t = 10d, 40d$ and $120d$ for both cases.

increase, which proves the stability of the fuzzy computational scheme.

CONCLUSIONS

The theory of fuzzy logic as well as the fuzzy partial differential equations under the generalized Hukuhara derivative, proved to be valuable tools in applications for groundwater physical problems which were characterized largely from ambiguities and uncertainties. Researchers and engineers can now solve practical problems including the vital factor of fuzziness in their decision making and planning.

Analyzing the results and taking into account the values of the transformed Hausdorff distance, it was concluded that the proposed fuzzy Crank-Nicolson procedure approach with satisfactory accuracy the fuzzy analytical results. Therefore, the proposed Crank-Nicolson scheme in a fuzzy environment is perfectly suited for scientific purposes and treatment of similar physical problems as that of this paper but also various other problems such as the enrichment of an underground aquifer by a ditch, etc.

Regarding the recharging and discharging cases the function $\tilde{h}(x, t)$ is [(ii)-p] differentiable with respect to x and [(i)-p] with respect to t . The function $\partial \tilde{h}(x, t) / \partial x$ is [(ii)-p] differentiable with respect to x .

It must always be borne in mind that the percentage of fuzziness must be wisely defined and under no circumstances frivolous and careless as it can lead to misleading results and wrong management decisions. The expertise and experience of the engineers or researchers can contribute to the appropriate estimation of the percentage of uncertainty.

In this work, an implicit finite difference numerical scheme was proposed to solve the linearized Boussinesq equation in a fuzzy environment. An extension of this research could be the solution with both analytical and numerical approach of the non-linear fuzzy equation of Boussinesq. Also, the solution of the tridiagonal system of the fuzzy numerical scheme could be improved in terms of the computing process speed, as in fuzzy environment more and complex calculations are required and the numerical schemes becomes costly in time.

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DATA AVAILABILITY STATEMENT

All relevant data are included in the paper or its Supplementary Information.

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