Calculation method for steady-state pollutant concentration in mixing zones considering variable lateral diffusion coefficient

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ABSTRACT

Prediction of the pollutant mixing zone (PMZ) near the discharge outfall in Huangshaxi shows large error when using the methods based on the constant lateral diffusion assumption. The discrepancy is due to the lack of consideration of the diffusion coefficient variation. The variable lateral diffusion coefficient is proposed to be a function of the longitudinal distance from the outfall. Analytical solution of the two-dimensional advection–diffusion equation of a pollutant is derived and discussed. Formulas to characterize the geometry of the PMZ are derived based on this solution, and a standard curve describing the boundary of the PMZ is obtained by proper choices of the normalization scales. The change of PMZ topology due to the variable diffusion coefficient is then discussed using these formulas. The criterion of assuming the lateral diffusion coefficient to be constant without large error in PMZ geometry is found. It is also demonstrated how to use these analytical formulas in the inverse problems including estimating the lateral diffusion coefficient in rivers by convenient measurements, and determining the maximum allowable discharge load based on the limitations of the geometrical scales of the PMZ. Finally, applications of the obtained formulas to onsite PMZ measurements in Huangshaxi present excellent agreement.

Key words | advection–diffusion equation, analytical solution, pollutant mixing zone, rivers

INTRODUCTION

The prediction of the mixing of pollutant in rivers and streams is a common civil and environmental application of the turbulent diffusion theory. A key problem is the determination of the diffusion coefficient. Fischer et al. (1979) collected over 70 sets of laboratory and field test data to study the lateral diffusion coefficient in a uniform straight channel. Their analysis indicates that the lateral diffusion coefficient in certain reaches of the channel/river may be treated as a constant. The value of the constant, however, is strongly affected by the river hydrology, hydraulic conditions, irregular terrain, etc. Therefore, studies during the past several decades were mostly committed to seeking the relation between the appropriate constant lateral diffusion coefficient and the mean hydraulic parameters of river reaches (Fischer & Hanamura 1975; Beltaos 1980; Webel & Schatzmann 1984). These studies show that it is proper to assume the lateral diffusion coefficient of the river to be constant, in most of the practical engineering cases, with acceptable error (Zhang & Li 1995; Lung 1995; IDEQ 2015). Many hydrology theories have been developed based on this assumption.

Under the condition of a constant lateral diffusion coefficient, considering the longitudinal advection and lateral diffusion in river as the major transport mechanisms, Li (1972) obtained the analytical solutions of the simplified two-dimensional advection–diffusion equation of the pollutant concentration. When the wastewater quality exceeds the environmental quality standards, a mixing zone of non-acutely toxic pollutant is acceptable as long as its area is relatively small (Rodríguez Benítez et al. 2016). One of the pollutant mixing zone (PMZ) technical handbooks, for instance, was proposed by the Idaho Department of Environmental Quality (IDEQ). The manual includes the PMZ setting-up rules, approval process, monitoring procedures, and water-quality modelling (IDEQ 2015). The geometric scales of the mixing zone in such handbooks...
and research are usually defined by contour-lines of the concentration field (at certain level as required by the water-quality standard) obtained by numerical simulation using two-dimensional water-quality models (Hamzeh 2016). The polynomial fitting equations of these contour-lines not only lack generality, but also may lead to wrong prediction when the flow parameters differ from their calibration ranges.

Wu & Jia (2009) and Wu et al. (2009, 2011) developed an analytical PMZ calculation method for the constant lateral diffusion coefficient condition in wide, rectangular and straight rivers. Based on the analytical solution of the simplified two-dimensional advection–diffusion equation by Fischer et al. (1979) and Li (1972), analytical formulas of the geometric scales and the mixing zone area were obtained. These formulas clearly demonstrate the constitutive relations between the flow parameters (such as the discharge rate, mean flow velocity, and lateral diffusion constant) and the macro geometric scales of the PMZ. Moreover, these analytical formulas build up a standard-curve equation describing the boundary of the PMZ. This standard curve formula is non-dimensionalized by certain characteristic scales, and is in a universal form which is valid for all the flow belonging to such a simplification category of rivers. The accuracy of these analytical formulas was evaluated by the data of Huang et al. (2006), which contains the water-environment-related indexes and the hydrology/water-quality pollutant load monitored in the Three Gorges of the Yangtze River in China. The predicted PMZ boundary curve agreed very well with the measured data near the discharging outfall of the Fuling phosphate fertilizer factory. Discrepancy, however, appeared in predicting the shape of the PMZ near the Huangshaxi municipal sewage outfall. We proposed that the error is due to the variation of the lateral diffusion coefficient during the transport of pollutant near the Huangshaxi municipal sewage outfall, which violates the constant assumption used during deriving the analytical solutions. A physical picture of such diffusion variation can be described: the shallows on the shore near the discharge outfall, where the diffusion of pollutant begins, are dominated by small-scale vortices which result in a small diffusion coefficient; as the diffusion cloud moved further from the bank at more downstream locations, the effects of the larger-scale vortices in the main flow of the river are enhanced; thus the diffusion coefficient becomes larger (David et al. 2013; Sinan 2014; Noori et al. 2015).

In this paper, the simplified two-dimensional advection–diffusion equation of a pollutant is solved to obtain the analytical solution of the pollutant concentration. The sewage outfall is considered to be constant, point-source, riverbank discharging. We first review the present analytical solutions under a constant diffusion coefficient assumption. Then a new model for the variable diffusion coefficient is proposed, followed by derivation of the analytical solution of pollutant concentration. Formulas describing the geometric scales, area, and the standard curve equation for the boundary of the PMZ are obtained by proper normalizing of the analytical concentration solution. By comparing these formulas with the previous ones with the constant diffusion coefficient assumption, the impact of lateral diffusion coefficient variation on the topology of the PMZ is discussed. Finally, these formulas are validated by two practical applications in natural rivers.

**ANALYTICAL CONCENTRATION DISTRIBUTION AND CHARACTERISTIC ANALYSIS**

**Previous studies**

Under steady-state, the leading convecting-type dynamics that determine the transport of the pollutant is the longitudinal (i.e., $x$ direction along the river flow) convection because $U \gg V$, where $U$ is the mean flow velocity and $V$ is the velocity in the lateral direction. In such situations, the two-dimensional advection–diffusion equation of the pollutant concentration $C(x,y)$ (with dimension $[M \, L^{-3}]$, where $M$ is unit of mass) in a river is (see Li (1972), Equation (30), p. 178):

$$U \frac{\partial C}{\partial x} = \frac{\partial}{\partial x} \left( E_x \frac{\partial C}{\partial x} \right) + \frac{\partial}{\partial y} \left( E_y \frac{\partial C}{\partial y} \right),$$

(1)

in which $x$ and $y$ (lateral direction perpendicular to the river bank and pointing to the opposite shore) are oriented from the discharge outfall with dimension $[L]$; $U$ is the mean flow velocity with dimension $[L \, T^{-1}]$, where $T$ is unit of time; $E_x$ and $E_y$ are the longitudinal and lateral diffusion coefficient with dimension $[L^2 \, T^{-1}]$, respectively.

For the diffusion far away from the sewage outfall (i.e., $x U \gg E_x$), the role of the diffusion in the $x$ direction is insignificant compared to the advection effect, therefore negligible (refer to Li (1972), pp. 179–180, for complete analysis). Then, Equation (1) can be simplified as:

$$U \frac{\partial C}{\partial x} = \frac{\partial}{\partial y} \left( E_y \frac{\partial C}{\partial y} \right).$$

(2)
To the best of our knowledge, no general analytical solution has been obtained for Equation (2) yet. When $E_y$ is a constant (i.e., $E_y = E_c$) and a constant point-source discharges at $x = 0$, $y = 0$, the analytical solution of Equation (2) is (see Li (1972), Equation (35), p. 181):

$$C(x, y) = \frac{m}{2H\sqrt{\pi E_c UX}} \exp\left(-\frac{Uy^2}{4E_cx}\right),$$  \hspace{1cm} (3)

where $(m/H)$ is the discharge mass per unit depth per time with dimension $[M \cdot T^{-1} \cdot L^{-1}]$; $H$ is the mean depth with dimension $[L]$. Equation (3) defines the set of iso-concentration lines as semi-elliptical curves (Wu & Jia 2009). Often in practical flows, however, significant discrepancy shows when Equation (3) is used to describe the concentration field. In our precursor study, it was seen that the normalized shape of the concentration field does not always agree with the contours defined by Equation (3). For example, a pear-shaped PMZ was observed in Huangshaxi, China. It has a sharp tip near the sewage outfall and a blunt end far away downstream. Note that including the constant longitudinal diffusion as in Equation (2) does not lead to a better prediction: the shape of the iso-concentration curve remains nearly symmetric about the centre of the field in the longitudinal direction (Wu et al. 2009).

**New analytical solutions of pollutant transport equation**

Such failure of current solutions are due to the nature of $E_y$: it varies with the flow rather than being a constant. To determine $E_y$ in Equation (2), recall that turbulence consists of vortices of various scales. Generally speaking, the large-scale vortices play the major role in the transport of momentum, mass and heat (Xia 1992). The longer the period of time after the pollutants have been discharged from the outfall, the further they get transported into the main stream of the river where the large-scale vortices are dominant. Therefore, the growth of the PMZ can be described as a function of either $x$ or $y$; i.e., the lateral diffusion coefficient increases with the range of the PMZ.

In this study, we assume that the river lateral diffusion coefficient is proportional to the diffusion distance $x$, borrowing the above physical picture and the analogy of the theories of atmospheric mixing (Li 2009). The variable lateral diffusion coefficient then is written as follows:

$$E_y(x) = \gamma_y x^{\alpha_y},$$  \hspace{1cm} (4)

in which $\gamma_y$ and $\alpha_y$ are positive constants. When $\alpha_y = 0$, Equation (4) represents the constant lateral diffusion coefficient scenario with $E_y = \gamma_y x^{\alpha_y}$ (subscript 0 denotes the constant diffusion coefficient case hereafter).

Substituting Equation (4) into Equation (2) gives:

$$U \frac{\partial C}{\partial x} = \gamma_y x^{\alpha_y} \frac{\partial^2 C}{\partial y^2}. \hspace{1cm} (5)$$

Arrange Equation (5) to get:

$$U \frac{\partial C}{\partial x} = \frac{\gamma_y}{1 + \alpha_y} \frac{\partial^2 C}{\partial y^2}. \hspace{1cm} (6)$$

Let $X = x^{(1 + \alpha_y)}$ and $E = \gamma_y/(1 + \alpha_y)$, Equation (6) transforms into the form of Equation (2), with a constant diffusion coefficient:

$$U \frac{\partial C}{\partial X} = \frac{\gamma_y}{1 + \alpha_y} \frac{\partial^2 C}{\partial y^2}. \hspace{1cm} (7)$$

Based on Equation (3) and considering the reflection by the riverbank that doubles the concentration on one side (see Fischer et al. (1979), p. 45, Equation (2.45)), the analytical solution of Equation (7) is:

$$C(X, y) = \frac{m}{H\sqrt{\pi EUX}} \exp\left(-\frac{Uy^2}{4EX}\right). \hspace{1cm} (8)$$

Putting the definition of $X$ and $E$ back into Equation (8) and simplifying it, we obtain the pollutant concentration distribution in rivers with a variable lateral diffusion coefficient as follows:

$$C(x, y) = \frac{m}{H\sqrt{\pi U}} \sqrt{\frac{1 + \alpha_y}{\gamma_y x^{(1 + \alpha_y)}}} \exp\left[-\frac{(1 + \alpha_y)Uy^2}{4\gamma_y x^{(1 + \alpha_y)}}\right]. \hspace{1cm} (9)$$

**Characteristic analysis of the concentration distribution**

Based on Equation (9), we then explore the impact of the variable lateral diffusion coefficient on the characteristics of the PMZ, by comparing the concentration fields of different values of $\alpha_y$. At each longitudinal location $x$ downstream of the discharge outfall, Equation (9) leads to the concentration distribution as:

$$C(x, y) = C_m(x) \cdot \exp\left[-\frac{y^2}{2\sigma_y^2(x)}\right], \hspace{1cm} (10)$$
in which \( C_m(x) = C(x, 0) \) is the maximum concentration in the lateral direction; \( \sigma_x^2(x) \) is the lateral variance of the concentration due to the lateral diffusion. Equations (9) and (10) indicate that the lateral concentration of the pollutant discharged from the riverbank exhibits a semi-normal distribution. The maximum concentration at every longitudinal location, \( C_m(x) \), happens at the riverbank \((y = 0)\). This maximum concentration is proportional to \((1 + \alpha_y)^{\frac{1}{5}}\), and decays as the \(-\frac{(1 + \alpha_y)}{2}\) power of the distance from the outfall. The lateral variance of the concentration is also a function of the longitudinal distance:

\[
\sigma_x^2(x) = \frac{2\gamma_x x^{(1 + \alpha_y)}}{U (1 + \alpha_y)} \quad (11)
\]

Recall that when \( \alpha_y = 0 \), it represents the constant lateral diffusion coefficient scenario. The corresponding maximum concentration and lateral variance are \( C_{m,0}(x) \) and \( \sigma_{x,0}^2(x) \).

From Equation (9), it can be obtained that the ratio of \( C_m(x) \) with the variable and constant lateral diffusion coefficients (while all other flow and discharge conditions stay the same), namely \( \lambda_m \) hereafter, is:

\[
\lambda_m \equiv \frac{C_m}{C_{m,0}} = \frac{\gamma_{y,0} (1 + \alpha_y)}{\gamma_y x^{\alpha_y}} \quad \text{for} \quad \alpha_y > 0. \quad (12)
\]

\( \lambda_m = 1 \) when \( \alpha_y = 0 \). Similarly, the ratio of \( \sigma_x^2(x) \) with the variable and constant lateral diffusion coefficients, namely \( \lambda_\sigma \) hereafter, is:

\[
\lambda_\sigma \equiv \frac{\sigma_x^2}{\sigma_{x,0}^2} = \frac{\gamma_y x^{\alpha_y}}{\gamma_{y,0} (1 + \alpha_y)} \quad \text{for} \quad \alpha_y > 0. \quad (13)
\]

\( \lambda_\sigma = 1 \) when \( \alpha_y = 0 \). Note that \( \lambda_\sigma = \lambda_m^{-1} \).

The longitudinal profiles of \( \lambda_m \) and \( \lambda_\sigma \) calculated by Equations (12) and (13) are plotted in Figures 1 and 2 with \( \alpha_y = 0.5, 1.0, \) and \( 2.0 \). The constant diffusion coefficient case is also shown for comparison. \( \gamma_y/\gamma_{y,0} \) is chosen to be 1/5 here; other values give similar results. \( \lambda_m \) decreases exponentially with the distance \( x \) as determined by Equation (12). This represents the physical picture that when the pollutant moves downstream, the mixing cloud grows and is more likely to be mixed by the large-scale turbulent vortices. The lateral diffusion coefficient increases by the large vortices and it decreases the maximum concentration. With increase in \( \alpha_y \), the change of \( \lambda_m \) near \( x = 0 \) becomes more rapid. A similar trend (reversed as an exponential increase) shows in the profiles of \( \lambda_\sigma \) (Figure 2). In all, considering the variable \( E_y(x) \), the pollutant is more concentrated near the discharge riverbank near the outfall (i.e., \( \sigma_y^2 \rightarrow 0 \) as \( x \rightarrow 0 \)). As the turbulent eddies grow downstream, the lateral diffusion is amplified and generates a significant wider mixing zone (e.g., \( \lambda_\sigma > 2 \) for \( x > 40, \alpha_y = 1.0 \) in Figure 2).

**GEOMETRIC CHARACTERISTICS OF PMZ**

The summation of the background concentration \( (C_b) \) and the concentration allowed to be increased by the discharge \( (C_a) \); subscript ‘a’ represents ‘allowed’ hereafter) should meet the requirement of the pollutant concentration in the mixing zone standard \( (C_{std}) \). The iso-concentration line of \( C_a = C_{std} - C_b \) defining the boundary of the PMZ is (refer to Equation (9)):

\[
C_a = \frac{m}{H \sqrt{\pi U}} \left( \frac{1 + \alpha_y}{\gamma_y x^{(1 + \alpha_y)}} \right) \exp \left[ -\frac{(1 + \alpha_y) U y^2}{4 \gamma_y x^{(1 + \alpha_y)}} \right]. \quad (14)
\]

In the following subsections, analytical formulas for the characteristic geometric parameters of the PMZ enclosed by the curve defined in Equation (14) will be derived and discussed.
Maximum length and width and maximum width location

Let \( y = 0 \) in Equation (14), the analytical formula for the maximum length of the PMZ is:

\[
L_s = \left( \frac{1 + \alpha_y}{\pi U_y HC_a} \right)^{2/(1 + \alpha_y)}.
\]  

(15)

Re-write Equation (14) as:

\[
C_s^{x(1+\alpha_y)/2} = \frac{m}{H \sqrt{\pi U}} \sqrt{\frac{1 + \alpha_y}{y}} \exp \left[ \frac{(1 + \alpha_y)Uy^2}{4\gamma y} \right] \cdot
\]

(16)

Taking the derivative of each term in both sides of Equation (16) and letting \( dy/dx = 0 \) gives:

\[
C_s^{x(1+\alpha_y)/2} = \frac{m}{H \sqrt{\pi U}} \sqrt{\frac{1 + \alpha_y}{y}} \exp \left[ \frac{(1 + \alpha_y)Uy^2}{4\gamma y} \right] \cdot
\]

(17)

Both sides of Equations (16) and (17) are equal; thus the longitudinal coordinate of the PMZ's maximum width, \( b_s \), is:

\[
L_c = \left[ \frac{(1 + \alpha_y)U}{2\gamma_y} b_s \right]^{1/(1+\alpha_y)}.
\]  

(18)

Substituting Equation (18) into Equation (14) (\( L_c \) as variable \( x \) and solve for \( y \)), and considering Equation (15), the maximum width of PMZ is:

\[
b_s = \sqrt[4]{\frac{2}{\pi e UHC_a}} = \sqrt[4]{\frac{2\gamma y L_s^{1+\alpha_y}}{(1 + \alpha_y)eU}}.
\]  

(19)

in which ‘e’ is the mathematical constant. Substituting Equation (19) into Equation (18), the longitudinal coordinate of the maximum width of the PMZ is related to its maximum length as:

\[
L_c = L_s e^{-1/(1 + \alpha_y)}
\]  

(20)

Standard curve equation of the boundary

When \( x \) is normalized by the maximum length and \( y \) by the maximum width of the PMZ, Equation (14) defining the boundary of the PMZ is in a universal form:

\[
\left( \frac{y}{b_s} \right)^2 = -e\left( \frac{x}{L_s} \right)^{(1+\alpha_y)} \ln \left( \frac{x}{L_s} \right)^{(1+\alpha_y)}.
\]  

(21)

Note that it is only a function of \( \alpha_y \). The model constant \( \gamma_y \), on the other hand, changes the length scales that are used for the normalization.

Area and area coefficient

Taking the definite integral of the square root of Equation (21) in the \( x \) direction from 0 to \( L_s \), the area of the PMZ is:

\[
S = \int_0^{L_s} b_s \left[ -e\left( \frac{x}{L_s} \right)^{(1+\alpha_y)} \ln \left( \frac{x}{L_s} \right)^{(1+\alpha_y)} \right] dx.
\]  

(22)

Substituting the variables as \( x/L_s = \zeta \) and then \( \eta = \zeta^{(1+\alpha_y)/2} \), it can be rewritten as:

\[
S = \frac{2}{5 + \alpha_y} \sqrt{\alpha L_s b_s} \int_0^{1} \sqrt{\ln \zeta^{-(1+\alpha_y)} d\zeta} \zeta^{(1+\alpha_y)/2}
\]

\[
= \frac{2}{5 + \alpha_y} \sqrt{\alpha L_s b_s} \int_0^{1} \sqrt{\ln \eta^{-1}} d\eta.
\]  

(23)

From a mathematical table: \( \int_0^{1} \sqrt{\ln \eta^{-1}} d\eta = \sqrt{\pi}/2 \), the analytical formula for PMZ area will be:

\[
S = \mu L_c b_s.
\]  

(24)

where the area coefficient

\[
\mu = \frac{\sqrt{\pi e}}{5 + \alpha_y} \sqrt{\frac{2(1 + \alpha_y)}{5 + \alpha_y}}
\]  

(25)

is a function of \( \alpha_y \). It decreases monotonically with the increasing of \( \alpha_y \). \( \mu_{\text{max}} = 0.795 \), when \( \alpha_y = 0 \), is exactly the same as the results of Wu & Jia (2009) for constant lateral diffusion coefficient cases.

DISCUSSION AND INVERSE PROBLEMS

Topology of the PMZ

Equation (21) indicates that the shape of the PMZ standard boundary is only determined by \( \alpha_y \). When the lateral diffusion coefficient is constant (i.e., \( \alpha_y = 0 \)), the standard curve
equation gives the result of Wu et al. (2011). The maximum width occurs at $L_c = L_0/e \approx 0.368 L_0$ (Equation (20)). When $\alpha_y = 0.443$, $L_c = 0.50 L_0$; that is, the widest section is in the middle of the PMZ. When $\alpha_y > 0.443$, the widest section moves further downstream towards the far end of the PMZ. Standard boundary curves at $\alpha_y = 0, 0.15, 0.50, 1.00$, and 2.00 are plotted in Figure 3. The location of the widest section (relative to $L_0$) and the area coefficient of the PMZ varying with $\alpha_y$ are plotted in Figure 4.

As can be seen from Figure 3, the shape of riverbank discharge PMZ changes significantly with $\alpha_y$. When $\alpha_y = 0$ (i.e., constant diffusion coefficient), the shape is blunt near the outfall and gradually becomes sharper downstream. As $\alpha_y$ increases, the region near the outfall becomes sharper, while the downstream end gets blunter due to the enhancement of diffusion by the increasing lateral diffusion coefficient. Figure 4 shows that the corresponding longitudinal coordinate of the maximum width, $L_c$, increases monotonically towards $L_0$ with the increase of $\alpha_y$. Figure 4 also shows that the area coefficient of the PMZ monotonically decreases with the increase of $\alpha_y$, from its maximum value of 0.795 at $\alpha_y = 0$ to 0.731 at $\alpha_y = 1$, and 0.499 at $\alpha_y = 4$.

**Inverse problem: estimate diffusion coefficient in rivers**

In practice, the lateral diffusion coefficient $E_y$ is hard to measure and usually assumed. The results of the current study, besides the advantage in characterizing the topology of the PMZ, can also be used to estimate the diffusion progress in the river. For example, if $L_c$ is observed to satisfy $L_0/e \leq L_c \leq (L_0/e + 5\%L_0)$ (i.e., $0.368 L_0 \leq L_c \leq 0.418 L_0$), $\alpha_y$ satisfies $0 \leq \alpha_y \leq 0.15$ (refer to Equation (20)). It can approximately be considered to be zero, and the lateral diffusion coefficient can be treated as a constant. The analytical formulas can even be used inversely to calculate the diffusion coefficient using the macro geometric scales which are much easier to obtain onsite. Here the authors propose two methods to estimate the lateral diffusion coefficient, in particular, the model constants $\gamma_y$ and $\alpha_y$, in the river by convenient onsite measurements.

**Dual cross-section method**

If the pollutant concentration distribution data is available at two sections $x_1$ and $x_2$, the variance, calculated according to its mathematical definition, can be used to calculate $\gamma_y$ and $\alpha_y$:

$$\sigma_{y,1}^2 = \frac{2 \gamma_y x_{11}^{1+\alpha_y}}{(1+\alpha_y)U} \quad \text{and} \quad \sigma_{y,2}^2 = \frac{2 \gamma_y x_{22}^{1+\alpha_y}}{(1+\alpha_y)U}.$$  

Then:

$$\left( \frac{\sigma_{y,1}}{\sigma_{y,2}} \right)^2 = \left( \frac{x_1}{x_2} \right)^{(1+\alpha_y)}$$  

and:

$$\alpha_y = 2 \ln \left( \frac{\sigma_{y,1}}{\sigma_{y,2}} \right) - 1.$$  

Finally, substituting the obtained $\alpha_y$ into Equation (26), $\gamma_y$ can be calculated as:

$$\gamma_y = \frac{(1 + \alpha_y)U\sigma_{y,1}^2}{2x_{11}^{1+\alpha_y}} \quad \text{or} \quad \gamma_y = \frac{(1 + \alpha_y)U\sigma_{y,2}^2}{2x_{22}^{1+\alpha_y}}.$$  

$E_y$ is then calculated by the obtained $\gamma_y$ and $\alpha_y$ based on Equation (4). Note that if $\alpha_y \leq 0.15$, the constant diffusion coefficient assumption can be employed and $\gamma_{y,0}$ is calculated by putting $\alpha_y = 0$ into Equation (29) to get $\gamma_{y,1}$ and $\gamma_{y,2}$. $E_y$ then is equal to the obtained $\gamma_{y,0} = (\gamma_{y,1} + \gamma_{y,2})/2$.

**Iso-concentration curve method**

In this method, Equation (20) is used reversely to calculate $\alpha_y$ by the measured $L_c$ and $L_0$. In practice, first, plot one
or several iso-concentration lines according to the onsite measurements of the two-dimensional pollutant concentration distribution. And then fit the mixing zone shape by adjusting $\alpha_y$ in the standard curve equation (Equation (21)). The maximum length ($L_c$), width ($b_s$) and the corresponding longitudinal coordinate ($L_s$) are obtained for each iso-concentration curve.

Using Equation (20), $\alpha_y$ can be calculated for each curve as:

$$\alpha_y = \ln^{-1}(L_s / L_c) - 1$$  \hspace{1cm} (30)

and $\gamma_y$ can be calculated based on the reverse of Equation (19) as (set $\alpha_y = 0$ if $\alpha_y \leq 0.15$):

$$\gamma_y = \frac{e(1 + \alpha_y)Ub^2}{2L_s^{0.5}(1 + \alpha_y)}.$$  \hspace{1cm} (31)

Averaging $\gamma_y$ and $\alpha_y$ over all the iso-concentration curves may increase the general accuracy.

### Calculation of the maximum discharge load

One more application of the above formulas is to determine the maximum discharge load, based on the mean flow velocity in the mixing zone $U$, the mean depth of mixing zone $H$, the variable lateral diffusion coefficient $E_y(x)$, the allowed concentration increases $C_n$, and the allowable range of the PMZ, i.e., the length [$L_s$], the width [$b_s$] or/and the area [$S$] ($\gamma$ denotes the allowable limit of scales hereafter).

The maximum discharge load limited by the allowable length [$L_s$], the width [$b_s$] and the area [$S$] are:

$$G_L = \sqrt{\frac{\pi y}{1 + \alpha_y}}HC_a[L_s^{1+\alpha_y}/2],$$  \hspace{1cm} (32)

$$G_b = \sqrt{\frac{\pi e}{2}}UHC_a[b_s],$$  \hspace{1cm} (33)

$$G_S = \sqrt{\frac{\pi}{1 + \alpha_y}}HC_a \left\{ \frac{\gamma_y}{1 + \alpha_y} U^{(2+\alpha_y)} \left( \frac{S}{2L_s^{0.5}} \right)^{1+\alpha_y} \left( \frac{1}{\gamma_y} \right)^{1/(3+\alpha_y)} \right\},$$  \hspace{1cm} (34)

respectively (refer to Equations (15), (19) and (24)). Taking the minimum value of the controlled pollutant load $G_L$, $G_b$ and $G_S$ gives the maximum pollutant load meeting all the requirements.

### APPLICATION EXAMPLES

#### Example I: Starch factory discharge in the Guangfu River

Figure 5 is the onsite observation of the mixing zone near the outfall of a starch factory in the Guangfu River (Wu et al. 2011). The maximum length and width of the PMZ, shown by the white high concentration region, are $L_s = 52$ m and $b_s = 5$ m, respectively. The corresponding longitudinal location of the maximum width is at $L_c = 12$ m. The mean flow velocity in the river is $U = 0.25$ m/s.

Using Equation (21) to fit the high concentration PMZ, $\alpha_y = 0.02 \approx 0$ is obtained for this case. It means that the lateral diffusion coefficient $E_y$ is constant in this region of the river. Thus, the boundary of the high concentration, white PMZ of this starch factory can be described by the standard curve equation (Equation (21)) as follows (in metric units):

$$\left( \frac{y}{S} \right)^2 = -e \left( \frac{x}{32} \right) \ln \left( \frac{x}{32} \right).$$  \hspace{1cm} (35)

The boundary curve defined by this formula is also plotted in Figure 5 as the dashed line (with the same projection angle as the observation), presenting good agreement with the onsite observation. The semi-elliptical shape of the PMZ is well represented. As the inverse problem discussed in the previous section, the lateral diffusion coefficient $E_y$ is constant in this region of the river. Thus, the boundary of the high concentration, white PMZ of this starch factory can be described by the standard curve equation (Equation (21)) as follows (in metric units):

$$\left( \frac{y}{S} \right)^2 = -e \left( \frac{x}{32} \right) \ln \left( \frac{x}{32} \right).$$  \hspace{1cm} (35)

The value we obtained here is, therefore, reasonable. Moreover,
our way of obtaining $E_y$ is much easier than the traditional method: one does not need the concentration profiles (for the iso-concentration curve method); instead, just a few length measurements (even from an onsite picture with proper scale) are sufficient to calculate $E_y$ using Equations (30) and (31).

**Example II: Municipal sewage discharge in Huangshaxi**

Figure 6 shows the contours of pollutant NH$_3$-N by field measurements around the municipal sewage mixing zone at Huangshaxi in the Yangtze River during a dry season (Huang et al. 2006). The shape of this PMZ is sharp near the sewage outfall and blunt at the far end. Compared with the solution with constant lateral diffusion coefficient (line marked $+$ in Figure 5), this shape (line marked $\times$ in Figure 3) is significantly different and the former solution totally fails in characterizing the PMZ in this case. The measured maximum length, maximum width and its corresponding longitudinal coordinate of maximum width with $\alpha_y = 1.67$ are compared in Table 1. It can be seen that the maximum relative errors between analytical fitted values and measured ones, among the three concentrations, are 4.0%, 4.2%, and 8.9%, for $L_s$, $b_s$, and $L_c$ respectively. The errors can be due to the possible local sudden change of the river topology, thus mild violation of the assumed form of diffusion coefficient variation defined in Equation (4). By putting $\alpha_y = 1.67$ into Equation (21), the iso-concentration curve equation of the municipal sewage mixing zone in Huangshaxi is obtained as (in metric units):

$$\left( \frac{y}{b_s} \right)^2 = -e^{\left( \frac{x}{L_s} \right)^{2.67}} \ln \left( \frac{x}{L_s} \right)^{2.67}.$$  

The three analytical curves are plotted on top of the measured field in Figure 6, based on their corresponding $L_s$ and $b_s$. Good consistency between the data and the analytical solution is exhibited. Note that the shape of the PMZ is significantly different from the one when $\alpha_y = 0$ (refer to Figures 3 and 5), because of the large $\alpha_y$. The region near the discharge outfall is much sharper, while the far end is much blunter. The analytical formulas derived in this study predicted well the topology of the pollutant, with small errors in the macro scales.

The mean flow velocity $U$ in the regions enclosed by the three iso-concentration curves were measured to be 0.75, 0.72 and 0.5 m/s, respectively. Then, applying the iso-concentration curve method proposed in the section ‘Iso-concentration curve method’ to solve the inverse problem, the mean $\gamma_y$ of the three curves is 0.0114. The maximum relative difference in $\gamma_y$ between the three is 3.4%, thus proving that the diffusion coefficient variation defined by Equation (4) is a reasonable assumption for this PMZ. We finally obtained the formula for the variable lateral diffusion coefficient of the sewage mixing zone in Huangshaxi as (in metric units):

$$E_y = \gamma_y x^{0.67} = 0.0114 \cdot x^{1.67}.$$  

**CONCLUSIONS**

We have proposed that the variable lateral diffusion coefficient in rivers is a function of the longitudinal distance
from the discharge outfall, and derived the analytical solution of the simplified two-dimensional advection-diffusion equation of pollutant in rivers. The aim of this work is to improve the prediction and characterization of the PMZ in the situations when assuming a constant lateral diffusion coefficient leads to significant errors. The solution is applicable to constant point-source discharge in straight, wide rivers/open-channels under steady-state. A set of analytical formulas describing the geometric features of the PMZ was derived from the analytical solution of pollutant concentration. We demonstrated that these formulas have the following advantages that benefit both future research and practical hydraulic applications: firstly, they reveal the constitutive relation linking the geometrical scales of the PMZ with the discharge and flow conditions; secondly, they are in a universal form that is able to describe the shape of the PMZ, ranging from semi-elliptical to pear-like, by a single model parameter; thirdly, they can be used in the inverse problems, including estimating the lateral diffusion coefficient in rivers by convenient measurements, and determining the maximum allowable discharge load based on the limitations of the geometrical scales of the PMZ. Further analysis explained the changes of the pollutant distribution with different variable lateral diffusion coefficients, and established a criterion of applying the constant lateral diffusion coefficient, that is, when the model constant $\alpha_l$ is less than 0.15. Validations of the current formulas using onsite data in a practical PMZ in rivers show excellent agreement between the measurements and the predictions.

ACKNOWLEDGEMENTS

This research is supported by the National Natural Science Foundation of China (51379097, 50979036).

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First received 18 December 2016; accepted in revised form 24 March 2017. Available online 10 April 2017